# Preventing Runs with Redemption Fees*


Xuesong Huang†
*Lingnan College*
*Sun Yat-sen University*

Todd Keister‡
*Rutgers University*


September 28, 2023


**Abstract**

We develop a model for evaluating policy proposals that aim to prevent runs on money market mutual funds (MMFs) and related intermediation arrangements. We first study policies that impose a redemption fee when the fund's liquid assets fall below a threshold level, similar to the reforms adopted in the U.S. in 2014. We show that such policies are often susceptible to a preemptive run in which investors rush to withdraw before the fees are imposed, in line with events at the onset of the Covid crisis in March 2020. We then study policies that impose a fee based on current redemption demand, even in normal times. We show that such policies are more effective at preventing runs, and we derive the best run-proof redemption fee policy. We show that this policy can have surprising features, such as setting the fee as a non-monotone function of redemption demand. Our framework indicates the new MMF reforms adopted in 2023 are an improvement over the previous round, but may still be susceptive to preemptive runs. We discuss the implications of our results for further reforms to MMFs and for stabilizing mutual funds more broadly.



*Preliminary and incomplete. An earlier draft of this paper was titled "Optimal Swing Pricing." We thank seminar participants at the Bank of Korea, Hitotsubashi University, Kyoto University, the National Graduate Institute for Policy Studies (GRIPS), Seoul National University and the University of Tokyo for helpful comments.



†135 Xingang Xilu, Guangzhou, Guangdong, China. Email: xuesong.huangll@gmail.com

‡Dept. of Economics, 75 Hamilton Street, New Brunswick, NJ 08901. E-mail: todd.keister@rutgers.edu

# 1 Introduction

The failure of Lehman Brothers in September 2008 sparked a run on prime money market mutual funds (MMFs) in the U.S, with over $400 billion dollars withdrawn in a two-week period. Because these funds play an important role in short-term funding markets, the U.S. Treasury and Federal Reserve introduced extraordinary programs to support MMF liquidity and provide guarantees to MMF investors. In 2014, the Securities Exchange Commission (SEC) introduced a set of reforms that aimed to prevent a repeat of this experience. These reforms allowed an MMF to limit redemptions and impose a redemption fee when the fund's liquid assets fell below a threshold level. Prime MMFs experienced heavy outflows again at the onset of the Covid crisis in March 2020, and the Federal Reserve again responded by providing extraordinary liquidity facilities. This episode is widely interpreted as evidence that the 2014 reforms were ineffective, and policymakers are again introducing reforms that aim to prevent runs on MMFs during future periods of financial stress. In July 2023, the SEC adopted new rules that replace the regime of redemption limits/fees based on a threshold for liquid assets with a plan for fees based on current redemption demand.[1] How effective this second set of reforms will be remains to be seen. At a conceptual level, however, the effectiveness of redemption fees as a financial stability tool, and the principles that should guide their use, are not well understood.

We develop a model for evaluating the effectiveness of different MMF redemption rules at preventing runs driven by investors' self-fulfilling beliefs. Our goal is to provide a framework for evaluating reform proposals and for understanding the principles that should govern MMF operations. We build on the approach in Engineer (1989), which adds an additional time period to the well-known framework of Diamond and Dybvig (1983). This additional period allows for the possibility that investors will run on the fund *preemptively* if they anticipate that redemptions may be restricted in future periods. Such preemptive reasoning is believed to have played an important role in the run on prime MMFs in March 2020.[2] We modify the model in several ways, in part to reflect the operational environment of a mutual fund rather than a bank. In particular, there is no first-come-first-served (or sequential service) constraint within a period. Instead, the fund is able to observe total redemption requests in each period before setting a redemption value for that period.

[1] The SEC's initial proposal in February 2022 would have required prime institutional funds to adopt a *swing-pricing* policy that adjusts the price of a share based on current redemption demand. We view the redemption fee policy in the SEC's final rule to be economically very similar to swing pricing.

[2] See, for example, the discussions in the reports of the President's Working Group on Financial Markets (2020) and the Securities Exchange Commission (2022).

We begin our analysis by using the model to study a policy that captures the spirit of the 2014 reforms. In this regime, the fund redeems shares at par until redemption demand is large enough that it would exhaust the fund's liquid assets if honored at par. When that happens, a redemption fee is imposed. We restrict the fee policy to be time-consistent, which prevents the fund from using non-credible threats in an attempt to influence investor behavior. We show that this policy often admits a run equilibrium. In this equilibrium, all investors with an opportunity to redeem in the first period do so because (*i*) they may need to redeem in the next period and (*ii*) they correctly anticipate that a redemption fee may be imposed at that point. In other words, when fees are imposed only once redemption demand is unusually high, investors may have an incentive to run preemptively on the fund. This discussion of why the 2014 reforms failed has focused largely on how the possibility a fund will restrict redemptions (impose a "gate") can lead to a preemptive run. Our results show that the other component of the 2014 rules, redemption fees, suffers from the same problem.

Our analysis also highlights the importance of information flows in determining the vulnerability of MMFs to a run. If the fund were able to detect a run right away, before any redemptions are processed, it could apply the fee to all redemptions. In that case, investors would be unable to withdraw preemptively and there would be no incentive to run on the fund. There are situations, however, where a run is underway but redemption demand initially remains within the normal range. Because the fund cannot yet distinguish these situations from normal times, it processes the initial redemption requests at par. This fact gives investors an incentive to try to redeem preemptively and opens the door to a run equilibrium.

We then turn our focus to policies that impose redemption fees based on current redemption demand, as in the 2023 reforms. To be effective in preventing runs, the fees must sometimes apply even when redemption demand is in the normal range. Such policies are costly because they impose risk on investors even in the absence of a run. However, an appropriate choice of such a fee policy can always eliminate the run equilibrium. We derive the best run-proof redemption fee policy and show how it balances the following tradeoff. On one hand, imposing a larger fee today leaves the fund in better condition in the future and thus reduces investors' incentives to redeem early. On the other hand, imposing a fee when redemption demand is normal harms investors who truly need liquidity, decreasing the value of participating in the fund. We show that the optimal fee for a given level of redemption demand depends on the relative likelihood of that demand in a run compared to normal times.

In a way, this result is very intuitive: the fee should be larger in situations that are more likely to occur in the event of a run and smaller in situations that are more likely in normal times. However, it can lead to unexpected patterns. Standard policy proposals satisfy a monotonicity property: as redemption demand increases, the fee charged to redeeming investors also increases, at least weakly. We show that, in the best run-proof policy, the fee may instead be a decreasing or non-monotone function of redemption demand. In part, this pattern reflects a desire to preserve the liquidity function of the fund, since applying a large fee in states where many investors truly need liquidity may be very costly.

After characterizing the optimal redemption fee policy, we use our framework to evaluate the new rules for prime institutional MMFs adopted by the SEC in July 2023. These rules require that, when a fund's redemption demand in a period exceeds 5% of its total assets, a redemption fee be imposed equal to the costs that would be associated with liquidating a pro-rata share of each asset in the fund's portfolio. This rule typically delivers lower welfare than the optimal policy we derive, but it has the advantage of being simpler. We examine this rule's effectiveness at preventing runs under two scenarios. When liquidation costs are currently elevated and expected to remain unchanged, the rule is effective: it does not admit a run equilibrium. However, if there is a significant chance that liquidation costs will increase in the following period, a run equilibrium often exists. These results indicate that an effective redemption fee policy must be *forward-looking* in the sense that the current fee must reflect beliefs about future liquidation costs. Otherwise, situations will arise in which investors expect the fee to increase over time, giving them an incentive to redeem preemptively.

Finally, we extend the analysis in three ways. First, we study the interaction between the fund's initial portfolio choice and the optimal design of the redemption fee policy. We show that the flexibility created by the optimal policy often makes holding excess liquidity undesirable. This result indicates that reform efforts should focus more on the design of the fees and less on requiring funds to hold larger liquidity buffers. We then study two variations in how the largest possible size of a run is modeled. In the baseline case, a known fraction of investors have an opportunity to redeem in the first pricing period, which implies that the maximum possible size of a run is also known in advance. We show that when this fraction is instead a random variable, the insights of our analysis remain largely unchanged. Finally, we take a robust-control approach in which the fund considers the worst-case scenario in terms of the size of the run. Specifically, after the fund chooses a redemption fee policy, nature chooses the size of a run to minimize the welfare associated with the chosen policy. In this case, we show that the optimal redemption fee policy is monotone: higher redemption demand is always associated with a (weakly) higher redemption fee.

**Related literature.** Our paper is related to several strands of the broad literature on preventing runs on financial intermediation arrangements. One strand of this literature studies swing pricing, that is, policies that adjust the price of a mutual fund share in response to redemption demand.[3] For example, Ma et al. (2022) use a Diamond-Dyvig framework to study a swing-pricing adjustment to a mutual fund's NAV that is an increasing function of the outflow. Interestingly, they show that swing pricing may improve liquidity provision by the fund in equilibrium since swing pricing can eliminate run incentives and therefore reduces the fund's need to hold liquid assets. The redemption fee policies we study are economically equivalent to a form of swing pricing. Similar to the result in Ma et al. (2022), we show that adopting the optimal run-proof redemption fee policy with no excess liquidity can yield higher welfare than having the fund hold excess liquidity. However, different from Ma et al. (2022), the optimal policy in our paper is not necessarily a monotone function of the redemption demand.

Lewrick and Schanz (2017b) study a Diamond-Dybvig model with an external asset market that incurs trading costs. They identify conditions under which the fund swings the NAV to maximize investors' welfare. In contrast, we study a different variation of the Diamond-Dybvig framework based on Engineer (1989) and fully characterize the best run-proof policy. In particular, we show how the optimal payment adjustment depends on the redemption demand. In a different framework with redemption demand triggered by an exogenous asset market shock, Capponi et al. (2020) study how the design of the swing pricing rule can break the negative feedback loop between mutual fund outflows and asset illiquidity. In contrast, we focus on the endogenous pattern of investors' redemption demand and study how the optimal rule balances the trade-off between preventing runs and maximizing investors' welfare.

Our paper is also related to the literature on preemptive banks runs under policies with deposit freezes and withdrawal fees, as in Engineer (1989), Cipriani et al. (2014); Cipriani and La Spada (2020) and Voellmy (2021). The logic underlying the preemptive runs in our framework is different, however. In those papers, the bank or fund processes redemption requests sequentially and halts redemptions when requests pass a threshold. If a run is underway, this action creates a backlog of investors with true liquidity needs who have not yet been served and who will redeem at the next available opportunity. This backlog implies that redemption demand will be high enough in the next period to trigger restrictions again,

[3] See Capponi et al. (2022) for a survey on this strand of literature.

which gives investors an incentive to redeem today if possible. In our model, in contrast, the fund processes all redemption requests in the period they are received – no backlog of unfilled requests is created. Instead, the existence of a preemptive run under the redemption fee policy is due to the fund's inability to detect a moderate-scale run today, which puts the fund in a bad position tomorrow. Furthermore, we show how a well-designed redemption fee policy can resolve this issue and prevent preemptive runs.

We also contribute to the literature on the mechanism design approach to bank runs pioneered by Wallace (1990). A defining characteristic of this strand of literature is that the bank/fund can consider a large set of contracts beyond the simple demand-deposit contract studied in Diamond and Dybvig (1983). In the classic three-period Diamond-Dybvig framework with sequential service and full commitment, efficient ways to prevent runs using direct mechanisms have been identified by Green and Lin (2003) and Huang (2023), while approaches using indirect mechanisms have been identified by Cavalcanti and Monteiro (2016) and Andolfatto et al. (2017).[4] We interpret the progression of policy reforms in recent years as expanding the set of contracts available to money market funds and thus bringing practice closer to the environments in these models. However, these papers all impose a sequential-service approach that is appropriate for banks but not for mutual funds. We remove the sequential service constraint while adding an additional consumption period to the model, which substantially changes the set of available contracts. We solve for the best run-proof contract and show how the result can be naturally interpreted as a policy with redemption fees based on current redemption demand. We show that an effective policy must be forward-looking in the sense that the fee depends on anticipated future market conditions. As a result, and different from Zeng (2017), the policy we identify can eliminate shareholder runs even with active fund liquidity management.

Finally, our paper is related to the empirical literature on mutual fund runs and patterns of redemptions. For example, Li et al. (2021) shows that the existence of a threshold for redemption fees and gates contributed to the run on money market mutual funds in March 2020, while Lewrick and Schanz (2017a) and Jin et al. (2022) document the effectiveness of swing pricing in removing the first-mover advantage in open-end mutual funds. We provide a unified theoretical framework that aims to help explain this evidence and points to more effective regulatory approaches. In particular, our results highlight a clear difference in effectiveness between redemption fees based on a fund's current liquid asset holdings (which reflect past redemption demand) and fees based on current redemption demand.

[4] For settings where the best direct mechanism does not prevent runs, see Peck and Shell (2003), Ennis and Keister (2009b) and Sultanum (2014), among others.

## 2 The model

Our model builds on Engineer (1989), which added an additional time period to the classic framework of Diamond and Dybvig (1983). We introduce a portfolio choice and fundamental uncertainty into this setting, and we modify the information structure to reflect the operation of a mutual fund rather than a bank. In this section, we describe the details of our environment and show that, despite these changes, the efficient allocation of resources is equivalent to that in the standard Diamond-Dybvig model. All proofs are in the appendix.

### 2.1 The environment

There are four time periods, $t = 0, 1, 2, 3$, and a single consumption good in each period. There are two technologies for transforming goods across periods, *storage* and *investment*. One unit of the good placed in storage in period 0 earns a gross return of 1 in any of the following periods. One unit invested in period 0 yields a return of $R > 1$ if held to maturity in period 3, but $r_t \leq 1$ if liquidated in period $t \in \{1, 2\}$. The value of $r_1$ is known, but $r_2$ is initially uncertain. We assume $r_2 \in \{\bar{r}, \underline{r}\}$, where $\bar{r} \geq r_1 > \underline{r}$ and $P(r_2 = \bar{r}) = q \geq 0$.

Each of a continuum of investors, indexed by $i \in [0, 1]$, is endowed with 1 unit of the good at $t = 0$ and has preferences:

$$u_i(c_1, c_2, c_3; \omega_i) = \begin{cases} u(c_1) & \text{if } \omega_i = 1 \\ u(c_1 + c_2) & \text{if } \omega_i = 2 \\ u(c_1 + c_2 + c_3) & \text{if } \omega_i = 3, \end{cases}$$

where $c_t$ is her consumption in period $t$ and $\omega_i$ is her liquidity-preference type. Both type-1 and type-2 investors are "impatient" in the sense that they need to consume before investment matures, while type-3 investors are "patient". We assume that a known fraction $\pi \in (0, 1)$ of investors will be impatient, but the distribution of these investors between type 1 and type 2 is random. In other words, there is no aggregate uncertainty about total early consumption demand, but there is uncertainty about its timing.[5] Let $\pi_1$ denote the fraction of investors who are type-1 and let $f(\pi_1)$ denote its probability density function for $\pi_1 \in [0, \pi]$. The fraction of type 2 investors is then $\pi - \pi_1$. Investors' types are private information, and each investor learns her own type gradually. In period 1, an investor discovers

[5] One interpretation of this assumption is that, based on historical data, the fund has a pretty good estimate of the daily liquidity needs among investors, but the actual timing of those liquidity needs, i.e., whether they occur in the morning or the afternoon, are less predictable.

only whether she is type 1 or not. In period 2, the remaining investors each discover whether they are type-2 or type-3.

In period 0, investors are able to pool their endowments in an arrangement that we call a *fund*. The fund uses these endowments to form a portfolio of storage and investment. Let $s$ denote the fraction of the initial endowment the fund places in storage; then $1-s$ denotes the fraction invested. After this point, investors are isolated from each other and can only interact with the fund.[6] In periods 1 and 2, investors receive updated information about their own type and may have an opportunity to submit a redemption request to the fund. We assume all type 1 investors can contact the fund in period 1, but only a fraction $\delta \in (0,1]$ of non-type-1 investors can do so. The remaining fraction $1-\delta$ are inattentive or otherwise unable to contact the fund in period 1. All investors who have not yet redeemed can contact the fund in periods 2 and 3.

The fund collects all redemption requests in a period and then allocates consumption to the redeeming investors. The operation of the fund is thus characterized by three functions. In period 1, the fund pays an amount $c_1(m_1)$ to each redeeming investor, where $m_1$ denotes the number of redemption requests. In period 2, the fund observes the number of new redemption requests $m_2$ and the realized liquidation value $r_2$, then pays $c_2(m_1, m_2, r_2)$ to these investors. Once an investor redeems her share in the fund, she immediately consumes and exits the economy. Each investor remaining in the fund in period 3 receives a pro-rata share of the fund's matured assets, which we denote $c_3(m_1, m_2, r_2)$.[7]

## 2.2 The efficient allocation

Suppose the fund were operated by a planner who could observe investors' preference types and choose when they redeem. This planner would clearly direct type $t$ investors to redeem only in period $t$, which implies redemption requests will satisfy $m_1 = \pi_1$ and $m_2 = \pi - \pi_1$. We can then write the payments $\{c_1(\pi_1), c_2(\pi_1), c_3(\pi_1)\}$ directly as functions of the state $\pi_1$. The planner would choose the fund's portfolio $(s, 1-s)$ and these functions $\{c_1, c_2, c_3\}$ to

[6] As in Wallace (1988) and others, this isolation assumption implies investors are unable to trade shares in the fund or other claims with each other in period 1.

[7] In practice, mutual funds and money market funds in particular often have sponsor supports, which have important implications for financial fragility as shown in Parlatore (2016). In our setup, we assume there is no sponsor support for the fund so that we can focus on the optimal design of the fund's payment schedule to eliminate financial fragility.

maximize investors' expected utility

$$\int_0^\pi [\pi_1 u(c_1(\pi_1)) + (\pi - \pi_1)u(c_2(\pi_1)) + (1-\pi)u(c_3(\pi_1))] f(\pi_1) d\pi_1$$

subject to the feasibility constraints

$$\begin{aligned} \pi_1 c_1(\pi_1) + (\pi - \pi_1)c_2(\pi_1) &= s \qquad \text{for all } \pi_1, \text{ and} \\ (1-\pi)c_3(\pi_1) &= R(1-s) \quad \text{for all } \pi_1. \end{aligned}$$

The constraints say that payments to impatient investors in both periods 1 and 2 will be made using goods placed in storage, while payments to patient investors in period 3 will be made using matured investment. Note that this plan is feasible because there is no uncertainty about the total number of impatient investors, and it implies that the efficient allocation is independent of the liquidation values $r_1$ and $r_2$.

The first-order conditions for $c_1$ and $c_2$ imply

$$u'(c_1^*(\pi_1)) = u'(c_2^*(\pi_1)) \quad \Rightarrow \quad c_1^*(\pi_1) = c_2^*(\pi_1) \qquad \text{for all } \pi_1,$$

that is, the planner always gives the same consumption to type 1 and type 2 investors. The feasibility constraints associated with a given $\pi_1$ can then be combined and simplified to

$$\pi c_1^*(\pi_1) + (1-\pi)\frac{c_3^*(\pi_1)}{R} = 1. \tag{1}$$

Using the first-order condition for $c_3$, we also have

$$u'(c_1^*(\pi_1)) = Ru'(c_3^*(\pi_1)). \tag{2}$$

Note that equations (1) and (2) are the same for all $\pi_1 \in [0, \pi]$, meaning the efficient allocation is the same in all states. Let $(c_E^*, c_L^*)$ denote the consumption given to impatient (type 1 and type 2) investors and to patient (type 3) investors in this allocation, respectively. Note that equation (2) and $R > 1$ imply $c_L^* > c_E^*$, meaning patient investors receive higher consumption. The planner's portfolio choice puts $\pi c_E^*$ in storage and invests $(1-\pi)c_L^*/R$. The following result summarizes this discussion.

**Proposition 1.** *Let $(c_E^*, c_L^*)$ denote the unique solution to equations (1) and (2). Then the first-best allocation gives $c_E^*$ to each type 1 consumer at $t = 1$ and to each type 2 consumer at $t = 2$, and it gives $c_L^*$ to each type 3 consumer at $t = 3$.*

Equations (1) and (2) also characterize the efficient allocation in a standard Diamond-Dybvig model with only two consumption periods. In other words, having an extra time period and uncertainty about the timing of early consumption demand do not change the efficient allocation of resources in our setting. The planner wants all impatient investors to consume $c_E^*$, regardless of whether they redeem in period 1 or 2, and wants all patient investors to consume $c_L^*$.

## 2.3 Discussion

**Sequential service.** Unlike the usual banking arrangement studied in Diamond and Dybvig (1983), Engineer (1989), and many others, the fund in our model does not need to serve redeeming investors one-at-a-time. Instead, it collects all redemption requests in a period before making any payments. The assumption matches the operation of an open-end mutual fund that pays redeeming investors only at the end of a *pricing period*, which often corresponds to the business day. In a standard three-period model, the optimal contract when there is no sequential service rules out bank runs under very general conditions (see Green and Lin, 2003, Section 3).[8] We show below that this result does not extend to the four-period model we study here. A key feature of this model is that investors redeeming in period 1 must be served by the end of that period, before the fund observes what will happen in period 2. In other words, even though investors are not paid one-at-a-time, a form of sequential service *across periods* arises naturally in any longer-horizon model, and this fact potentially opens the door to self-fulfilling runs.

**Liquidation costs.** Money market funds hold assets that are fairly liquid most of the time but may become less liquid in periods of financial stress. It is probably no coincidence that the runs observed in 2008 and 2020 both occurred during periods of significant stress and lower market liquidity. In the analysis below, we consider scenarios where the fund's investment is liquid ($r_1 \approx 1$) and where it is illiquid ($r_1 < 1$). In both cases, we show that investors' expectations about the future liquidation value ($r_2$) play an important role in shaping redemption behavior. We allow $r_2$ to be random to capture situations where investors are concerned that market conditions may deteriorate in the intermediate period.

Throughout the analysis, we assume the liquidation values $r_t$ are independent of the fund's liquidation choices. In other words, we assume the fund is a relatively small player in the market for those securities. It may be interesting to extend our analysis to situations

[8] See also the discussion in Andolfatto and Nosal (2020).

where the fund recognizes that its own sale of assets may drive down the market price.[9]

**Partial runs.** Our assumption that only a fraction $\delta$ of non-type-1 investors can redeem at $t = 1$ captures the idea that a run does not typically take place within a single day or pricing period.[10] Instead, a run is typically spread over time, which makes identifying a run difficult in the early stages. If $\delta = 1$, a run will involve all of the fund's investors requesting redemption in the first period and, hence, is easily identified before the fund makes any payments to investors. When $\delta$ is below 1, in contrast, a run at $t = 1$ is partial, with only some investors participating. In that situation, the fund may initially be unsure whether the observed redemption demand is fundamental (with a large realization of $\pi_1$) or instead reflects a run. We show below such uncertainty is necessary for self-fulfilling runs to be possible in this setting. For now, we assume $\delta$ is a known constant. In Section 5.1, we extend the model to introduce uncertainty about the size of $\delta$.

# 3 Preemptive runs

We now return to the setting where investors' types are private information. In this section, we study equilibrium when the fund aims to implement the first-best allocation described in Proposition 1. We first describe the payment functions $\{c_1, c_2, c_3\}$ used by the fund and argue they match key features of the rules for MMFs adopted in the U.S. in 2014. We then study the resulting withdrawal game played by investors. This game implements the first-best allocation by design, but we show that a run equilibrium often also exists. The runs that occur are *preemptive* in the sense that non-type 1 investors are withdrawing in period 1 because they worry that $(i)$ a fee will be imposed in period 2 and $(ii)$ they may need to redeem in period 2. Their best response is then to redeem in period 1 in an attempt to exit the fund before the fee is imposed.

## 3.1 Contracts and the redemption game

Because investors' types are private information, the fund allows investors to choose when to redeem their shares. In general, a contract specifies a portfolio choice as well as payments to redeeming investors in each period as functions of the information available to the fund. In this section and the next, we assume the fund follows the planner's portfolio choice, $s = \pi c_E^*$.

[9] See section 6 for more detailed discussion.

[10] Chen et al. (2010) and Zeng (2017) use similar assumptions to capture the possibility of some investors being inactive.

A contract is then summarized by a collection of payment functions $c_1(m_1)$, $c_2(m_1, m_2, r_2)$, and $c_3(m_1, m_2, r_2)$. As the notation indicates, the payment $c_1$ made to redeeming investors in period 1 must be made before the fund observes $m_2$ (the number of withdrawal requests in period 2) or $r_2$ (the liquidation value of investment in that period).

**Feasibility.** A set of payment functions is feasible if the fund can generate these payments from its given asset portfolio. In period 1, the payment $c_1(m_1)$ must satisfy

$$m_1 c_1 + e_1 = s + r_1 \ell_1 \quad \text{for all } m_1, \tag{3}$$

where $e_1 \in [0, s]$ is the amount of storage held until period 2 ("excess liquidity") and $\ell_1 \in [0, 1-s]$ is the amount of investment liquidated in period 1. In period 2, feasibility requires that the function $c_2(m_1, m_2, r_2)$ satisfy

$$m_2 c_2 + e_2 = e_1 + r_2 \ell_2 \quad \text{for all } (m_1, m_2, r_2), \tag{4}$$

where $e_2 \in [0, s - e_1]$ is the amount of storage held until period 3 and $\ell_2 \in [0, 1 - s - \ell_1]$ is the amount of investment liquidated in period 2. Finally, feasibility in period 3 requires

$$(1 - m_1 - m_2) c_3 = R\,(1 - s - \ell_1 - \ell_2) + e_2 \quad \text{for all } (m_1, m_2, r_2). \tag{5}$$

Taken together, the payment functions $\{c_1, c_2, c_3\}$ are feasible if, for every $(m_1, m_2, r_2)$, there exist portfolio management choices $\{e_1, \ell_1, e_2, \ell_2\}$ such that equations (3) - (5) are satisfied.

**First best.** In this section, we assume the fund chooses the payment functions with the objective of implementing the efficient allocation characterized in Proposition 1. Implementing this allocation requires that the contract satisfies

$$c_1(m_1) = c_E^* \quad \text{for all } m_1 \leq \pi, \tag{6}$$

$$c_2(m_1, m_2, r_2) = c_E^* \quad \text{for all } m_1 + m_2 \leq \pi, \text{ and} \tag{7}$$

$$c_3(m_1, m_2, r_2) = c_L^* \quad \text{for all } m_1 + m_2 \leq \pi. \tag{8}$$

These payments are feasible when the fund sets

$$\begin{aligned} e_1 &= (\pi - m_1) c_E^* \qquad &&\text{and} \qquad \ell_1 = 0 \qquad &&\text{for } m_1 \leq \pi \\ e_2 &= 0 &&\text{and} \qquad \ell_2 = 0 &&\text{for } m_1 + m_2 \leq \pi. \end{aligned}$$

In other words, as long as total redemption demand does not exceed the fraction of impatient investors ($\pi$), the fund uses its portfolio to make payments exactly as the planner would. In particular, each redeeming investor is paid $c_E^*$ out of goods in storage and no investment is liquidated.

It is straightforward to show that the withdrawal game generated by any contract $\{c_1, c_2, c_3\}$ satisfying equations (6) - (8) has an equilibrium that implements the first-best allocation. In this equilibrium, only type 1 investors redeem in period 1, so $m_1 = \pi_1$, and only type 2 investors redeem in period 2, so $m_2 = \pi - \pi_1$. Investment is never liquidated and, therefore, the values of $(r_1, r_2)$ do not affect the allocation. The fact that $c_L^* > c_E^*$ implies that, in this equilibrium, non-type-1 investors have a strict incentive to wait in period 1 and type-3 investors have a strict incentive to wait in period 2. Because investors are small, unilateral deviations from equilibrium play do not change the fractions $(m_1, m_2)$. As a result, the payments in the contract associated with redemption demand greater than $\pi$ have no effect on individual investors' incentives, and this no-run equilibrium exists regardless of how those payments are specified.

**Time consistency.** Our interest is in studying whether the fund is fragile in the sense that another equilibrium exists in which investors rush to redeem at the first opportunity. The answer to this question depends crucially on the payments in the contract associated with levels of redemption demand greater than $\pi$. Diamond and Dybvig (1983) and others have shown that promising a "tough" response to high withdrawal demand can prevent a bank run equilibrium from existing in a three-period model. A similar result can be shown to hold here, although the form of the "tough" response is different. Consider a contract that, for all $m_1 > \pi$, sets $c_1 = 0$ and $e_1 = s$. In other words, if the fund detects a run is underway in period 1, all investors who have redeemed their shares will receive nothing in return. If $m_1 \leq \pi$ but $m_1 + m_2 > \pi$, meaning that the fund detects a run is underway only in period 2, the contract allocates the funds remaining resources so that $c_2$ and $c_3$ are strictly positive and satisfy $c_2 < c_3$. If a non-type-1 investor expects some other investors to run, she knows that $m_1$ will be greater than $\pi$ with positive probability. As long as consuming zero is sufficiently unattractive, she will strictly prefer to wait, so there cannot be an equilibrium where investors run in period 1. The fact that the contract sets $c_2 < c_3$ for all $(m_1, m_2)$ implies there cannot be an equilibrium where investors run in period 2. Therefore, this type of contract implements the first-best allocation without introducing the possibility of a run on the fund.[11]

However, the threat to give investors nothing in exchange for their shares in period 1 would clearly not be time-consistent. Once the redemption requests have been submitted, the fund would have a strong incentive to change course and offer positive consumption to all investors.[12] Our goal in this analysis is to provide policy advice for reforming MMFs, and we do not want such advice to rely on non-credible threats to punish investors in the event a run is detected. For this reason, we require that the contract offered by the fund be *time consistent* in the following sense: whenever total redemption demand exceeds $\pi$, making it clear that a run is underway, the fund must choose payments that maximize the sum of investors' utilities conditional on the given redemption demand, that is,

$$m_1 u(c_1) + \mathbb{E}\left[m_2 u(c_2) + (1 - m_1 - m_2) u(c_3) \mid m_1\right]. \tag{9}$$

In other words, the fund must always act in the best interests of its investors, even when a run is underway. Formally, we require that the payment functions $\{c_1, c_2, c_3\}$ satisfy the following two conditions.

(TC1) For $m_1 \leq \pi$ and $m_1 + m_2 > \pi$, $\{c_2, c_3\}$ must maximize equation (9) subject to the constraints in equations (4) - (5) with $c_1 = c_E^*$, $e_1 = s - m_1 c_E^*$, and $\ell_1 = 0$.

(TC2) For $m_1 > \pi$, $\{c_1, c_2, c_3\}$ must maximize equation (9) subject to constraints in equations (3) - (5) with $m_2 = \pi + \delta(1 - \pi) - m_1$.

Condition (TC1) applies to the scenario where it becomes apparent that a run is underway only in period 2. In this case, the $m_1$ investors who redeemed in period 1 have already each been paid $c_E^*$ out of goods held in storage. Time consistency requires that the remaining asset portfolio be used efficiently to make payments to the $m_2$ investors redeeming in the current period and the $1 - m_1 - m_2$ investors who will redeem in the final period. If the solution to this maximization problem sets $c_2 < c_E^*$, we will say the fund applies a *redemption fee* of $c_E^* - c_2$ in period 2.

[11] Note that the "tough" policy here is very different from suspending convertibility of shares, which Engineer (1989) showed is ineffective at preventing runs in a four-period model. When convertibility is suspended, investors who are unable to redeem in period 1 are able to try again in period 2. A rush to redeem in period 1 thus creates a backlog of redemption requests, which implies period-2 requests will be large as well. The contract we describe here, in contrast, honors all redemption requests in period 1 but may set the redemption price to zero. This policy is effective because it punishes redemption requests more heavily during a run in period 1, while always making future redemption more attractive.

[12] Alternatively, one could imagine investors who receive zero for their redeemed share might take legal action against the fund. The court system might then intervene to overrule the tough response to a run, as discussed by Ennis and Keister (2009a) in the banking context.

Condition (TC2) applies to the scenario where redemption demand in the first period is large enough to indicate a run is underway. In this case, the fund may choose to impose a redemption fee in both periods 1 and 2. Choosing the redemption fee for period 1 to maximize investors' expected utilities requires forecasting redemption demand in period 2. It is straightforward to show that, in this setting, an investor who turns out to be type 3 will never have an incentive to redeem in period 2. The fund will, therefore, accurately forecast that only those investors who were inattentive in period 1 and turn out to be type 2 will redeem in period 2, that is, $m_2 = (1-\delta)(\pi - \pi_1) = \pi + \delta(1-\pi) - m_1$. Time consistency requires that the fund act to maximize investors' utilities given the observed redemption demand $m_1$ and this forecast for $m_2$.

**Interpretation.** Taken together, the requirement that the fund $(i)$ follows the planner's allocation when redemption demand is below $\pi$ [equations (6) - (8)] and $(ii)$ satisfies the time-consistency constraints (TC1) - (TC2) when redemption demand is above $\pi$ fully determine the payment functions $\{c_1, c_2, c_3\}$. In other words, there is a unique contract that both implements the first-best allocation as an equilibrium and satisfies time consistency. We interpret this contract as capturing some key features of the reforms to prime MMFs that were adopted in the U.S. in 2014.[13] Under those rules, funds would redeem shares at par unless high redemption demand pushed the fund's liquid assets below a threshold level. Once this threshold was passed, a fund had the ability to impose a redemption fee of up to 2% and was directed to do so if it was deemed to be in the best interests of shareholders.[14] The threshold was set so that it would be hit only in extraordinary circumstances, not in normal times. We interpret this policy as attempting to rule out runs by imposing redemption fees that lie off the path of play in the no-run equilibrium, as required of our contract in equations (6) - (8). We interpret our time consistency constraints as capturing the spirit of the directive that the fund act in its shareholders' best interests in setting the fees. In the next subsection, we investigate whether these rules are successful in preventing runs in our model. We show the answer is 'no'.

[13] The full text of the 2014 rules is available at https://www.sec.gov/files/rules/final/2014/33-9616.pdf.

[14] The 2014 rules also allowed funds to impose redemption gates (that is, to suspend convertibility) once the threshold was passed. The approach in Engineer (1989) can be adapted to show that such a suspension policy is ineffective at preventing runs in our setting. For this reason, we focus our analysis on the more promising part of the 2014 rules: allowing redemption fees.

## 3.2 Equilibrium runs

We now show that a run equilibrium can exist when the fund attempts to implement the first-best allocation. To simplify the analysis, we specialize to log utility, $u(c) = \ln(c)$, in which case the the planner's allocation satisfies

$$c_1^*(\pi_1) = c_2^*(\pi_1) = 1, \quad c_3^*(\pi_1) = R, \quad \text{and} \quad s^* = \pi.$$

We also assume $r_1 = \bar{r} \leq 1$. In other words, we assume the liquidation value of investment will not improve but may worsen. The parameters $\underline{r}$ and $q$ then summarize investors' period-1 expectations about liquidation costs in period 2. If $\underline{r} = r_1$ or $q = 1$, liquidity conditions are expected to remain unchanged. If $q < 1$ and $\underline{r} < 1$, however, there is concern that investment will be more illiquid in the following period. In addition to simplifying calculations, these assumptions facilitate the interpretation of the model as the planner pays each impatient investor at "par" and gives each patient investor the return of the long-term investment.

**Time consistent payments.** We begin by deriving properties of the time-consistent payment schedules under these simplifying assumptions.

**Proposition 2.** *Condition (TC1) requires that, when $m_1 \leq \pi$ and $m_1 + m_2 > \pi$, the fund sets $e_2 = 0$ and sets*

$$c_2(m_1, m_2, r_2) = \max\left\{\frac{\pi - m_1}{m_2}, \frac{r_2(1-\pi) + \pi - m_1}{1 - m_1}\right\}$$
$$c_3(m_1, m_2, r_2) = \min\left\{\frac{R(1-\pi)}{1 - m_1 - m_2}, \frac{R(1-\pi) + \frac{R}{r_2}(\pi - m_1)}{1 - m_1}\right\}$$

Depending on redemption demand and the realized liquidation value of investment, the fund may choose to meet period 2 redemption requests only using goods in storage or it may choose to liquidate some investment. Either way, the time-consistent policy imposes a redemption fee ($c_2 < 1$) unless there is no cost of liquidating investment ($r_2 = 1$).

Next, we derive the implications of (TC2), which deals with the scenario where redemption demand in period 1 indicates that a run is underway. Given any portfolio at the beginning of period 2, the proof of Proposition 2 (in the appendix) shows the fund will choose $c_3(m_1, m_2, r_2) \geq c_2(m_1, m_2, r_2)$. Therefore, there is no run in period 2, and only type-2 investors will choose to redeem in that period.[15] Furthermore, when $m_1 > \pi$, the

[15] To simplify the presentation, we assume investors wait to redeem in cases where they are indifferent between redeeming and waiting.

fund knows that a run is underway, but forecasting redemption demand in future periods requires having a theory of what fraction of investors have participated in the run. We assume the fund believes that a run involves all attentive investors, which is correct in the equilibrium we study. Since $\pi$ and $\delta$ are known, the fund can infer from $m_1$ what redemption demand will be in the subsequent periods. Specifically, redemption demand in period 2 will come only from the type 2 investors who were inattentive in period 1, which implies $m_2 = \pi + \delta(1-\pi) - m_1$. Redemption demand in period 3 will come from type 3 investors who were inattentive in period 1, so $m_3 = (1-\delta)(1-\pi)$.[16] In other words, the fund is forward-looking and accurately forecasts future redemption demand $(m_2, m_3)$ using the observed current redemption demand $m_1$ and the assumption of equilibrium behavior. The following proposition characterizes the fund's optimal reaction to this situation.

**Proposition 3.** *Condition (TC2) requires that, when $m_1 > \pi$, the fund sets $e_2(r_2) = \ell_2(r_2) = 0$ and sets*

$$c_1 = c_2 = r_1(1-\pi) + \pi$$
$$c_3 = R(1-\pi) + \frac{R}{r_1}\pi \quad \textit{for any } r_2.$$

When redemption demand in period 1 indicates that a run is underway and the fund anticipates the liquidation value of investment may be lower in the future, it actively rebalances its portfolio and liquidates investment in period 1 to avoid the possibility of more costly liquidation in period 2. With no liquidation in period 2, the optimal payment in period 2 is independent of $r_2$ and the fund equalizes the payments in both middle periods.[17]

With these properties of the time-consistent contract in hand, we are ready to study equilibrium withdrawal behavior.

**The redemption game.** The payment functions $\{c_1, c_2, c_3\}$ induce a redemption game played by investors. Our interest is in whether there exists a run equilibrium in this redemption game.[18] As discussed above, type 3 investors who remain in period 2 will never have an incentive to redeem early. If a run occurs, therefore, it will be *preemptive* in the sense that it takes place only in period 1.

[16] In section 5.1, we show how this analysis changes when the fund is uncertain about the size of $\delta$.

[17] This portfolio rebalancing is reminiscent of the results in Zeng (2017), but reflects a very different motive. In that setting, the fund may preemptively liquidate investment even though doing so is costly because it wants to maintain a particular ratio of liquid to illiquid assets. Here, in contrast, the fund is preemptively liquidating investment because it recognizes it will need more liquid assets next period and worries that liquidation costs may increase in the meantime.

[18] We use perfect Bayesian equilibrium as the solution concept throughout the analysis.

To see if such an equilibrium exists, consider a non-type-1 investor who can redeem in period 1. Suppose she expects all other attentive non-type-1 investors to redeem in period 1, meaning redemption demand will be $m_1 = \pi_1 + \delta(1-\pi_1)$. There are two distinct possibilities. If $\pi_1$ is sufficiently small

$$\pi_1 \leq \frac{\pi - \delta}{1 - \delta}, \tag{10}$$

then $m_1 < \pi$ will hold. In this case, the run will initially be undetected and the fund will pay redeeming investors at par in period 1. However, the early redemptions by non-type-1 investors imply that the fund will experience higher than expected redemption demand in period 2, with $m_2 = (1-\delta)(\pi - \pi_1)$. The fund will then realize a run is underway and impose redemption fees in line with Proposition 2. Denote the payments in periods 2 and 3 in this case by $c_2(\pi_1, \delta; r_2)$ and $c_3(\pi_1, \delta; r_2)$, respectively.

The second possibility is that the inequality in (10) is reversed, in which case $m_1$ will be larger than $\pi$. The fund will then detect the run right away and impose redemption fees in line with Proposition 3. Combining these two cases, we can write a non-type-1 investor's expected payoff of redeeming and waiting as

$$\begin{aligned}
\text{Redeem:} \quad & \int_0^{\frac{\pi-\delta}{1-\delta}} u(1) f_n(\pi_1) d\pi_1 + \int_{\frac{\pi-\delta}{1-\delta}}^{\pi} u\left(r_1(1-\pi) + \pi\right) f_n(\pi_1) d\pi_1 \\
\text{Wait:} \quad & \int_0^{\frac{\pi-\delta}{1-\delta}} \{p_{\pi_1}[q u(c_2(\pi_1, \delta; \bar{r})) + (1-q) u(c_2(\pi_1, \delta; \underline{r}))] \\
& + (1 - p_{\pi_1})[q u(c_3(\pi_1, \delta; \bar{r})) + (1-q) u(c_3(\pi_1, \delta; \underline{r}))]\} f_n(\pi_1) d\pi_1 \\
& + \int_{\frac{\pi-\delta}{1-\delta}}^{\pi} \left[ p_{\pi_1} u\left(r_1(1-\pi) + \pi\right) + (1 - p_{\pi_1}) u\left(R(1-\pi) + \frac{R}{r_1}\pi\right)\right] f_n(\pi_1) d\pi_1.
\end{aligned}$$

Here, for each $\pi_1$, $p_{\pi_1} = \frac{\pi - \pi_1}{1 - \pi_1}$ is the probability of investor $i$ being type 2 in period 2 conditional on him being non-type-1 in period 1. The function $f_n$ represents the density of $\pi_1$ conditional on depositor $i$ being non-type-1. Using

$$F_n(x) = P(\pi_1 \leq x | \text{ Non-type-1 }) = \frac{P(\text{Non-type-1, } \pi_1 \leq x)}{P(\text{ Non-type-1 })} = \frac{\int_0^x (\pi - \pi_1) f(\pi_1) d\pi_1}{\int_0^\pi (\pi - \pi_1) f(\pi_1) d\pi_1},$$

this conditional density function is given by

$$f_n(x) = \frac{(\pi - x) f(x)}{\int_0^\pi (\pi - z) f(z) dz}.$$

If the investor chooses to redeem, she will be paid at par if $\pi_1$ is small enough to satisfy (10) and will otherwise be charged a redemption fee based on the current liquidation value $r_1$.If she instead chooses to wait, there are several possibilities. If $\pi_1$ is small, the run will be detected only in period 2 and she will be charged a fee that depends on the period in which she redeems and the realized period-2 liquidation value as determined in Proposition 2. If $\pi_1$ is large, however, the run will be detected in period 1 and the fund will do whatever liquidation is needed right away at $r_1$.In this case, the investor's consumption will be $c_2$ or $c_3$ as given in Proposition 3.

If the investor knew $\pi_1$ would be large enough to reveal the run in period 1, she would strictly prefer to wait because the redemption fee is the same in periods 1 and 2 and waiting gives higher consumption if she turns out to be type 3. In other words, if the fund could always detect a run right away, there would be no incentive to join the run since the time-consistent redemption fees take account of the liquidation costs imposed by early redemptions and, as a result, always give more consumption to investors who wait.

The potential incentive to join a run comes from the case where $\pi_1$ is small enough that the inequality in (10) holds. In this case, the run will go undetected in period 1 and redeeming investors are paid at par. This fact places the fund in a worse position in period 2, when redemption demand reveals the run and the fund imposes a redemption fee. If the liquidation value of investment in period 2 turns out to be low, the fund will impose a higher redemption fee in response, which makes it worse to be type 2. Furthermore, the liquidation of investment in period 2 also makes it worse to be type 3. Therefore, a non-type-1 investor may have an incentive to preemptively redeem in period 1. To illustrate this point, we present a series of examples that show how a preemptive run equilibrium can exist for a range of parameter values.

**Example 1:** Let $\pi = 0.6, R = 1.03, r_1 = \bar{r} = 1, \underline{r} = 0.7, q = 0.5$. We consider four probability distributions for $\pi_1 \in [0, \pi]$:

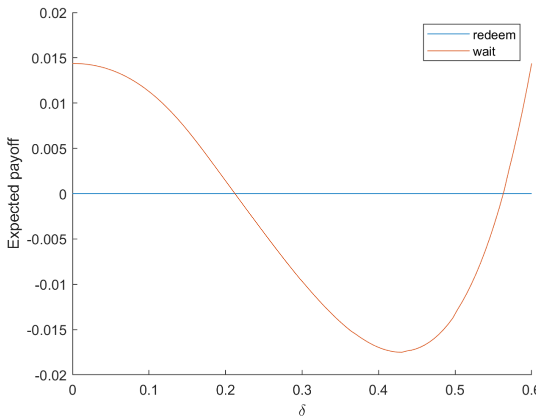
(a) A truncated normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.3$;

(b) A truncated normal distribution with mean $\mu = 0.3$ and standard deviation $\sigma = 0.3$;

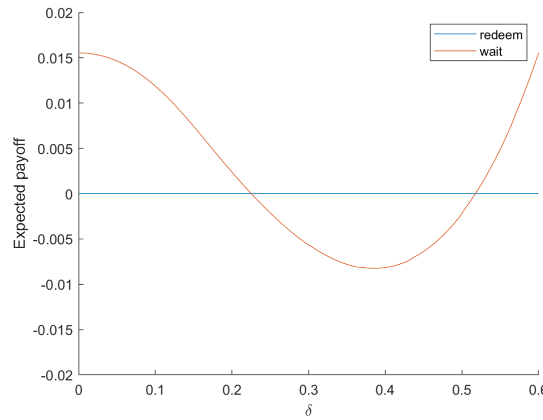
(c) A truncated normal distribution with mean $\mu = 0.5$ and standard deviation $\sigma = 0.3$;

(d) A uniform distribution on $[0, \pi]$.

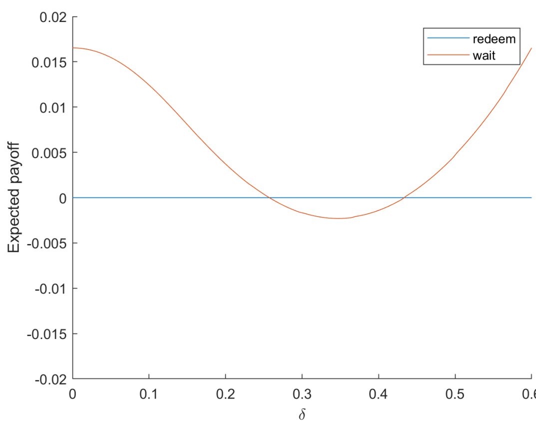
In this example, there is no cost of liquidating investment in period 1. Figure 1 depicts the expected payoffs of redeeming and of waiting for each of the four different probability distributions. Because $r_1 = 1$, an investor who redeems in period 1 is always paid at par, regardless of redemption demand in that period. This leads to a payoff of $\ln(1) = 0$ as indicated by the blue lines in each panel.



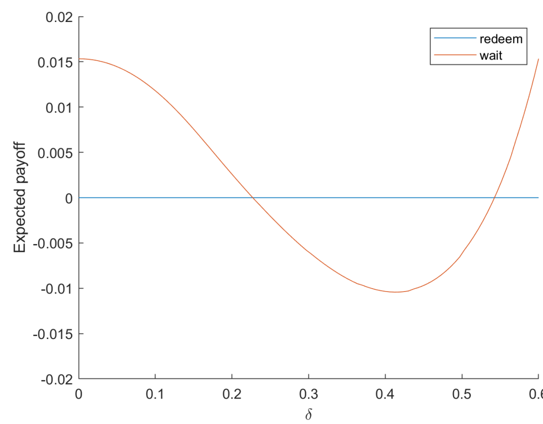

(a) Truncated normal with $\mu = 0$ and $\sigma = 0.3$




(b) Truncated normal with $\mu = 0.3$ and $\sigma = 0.3$




(c) Truncated normal with $\mu = 0.5$ and $\sigma = 0.3$




(d) Uniform on $[0, \pi]$

Figure 1: Fragility region of redemption fees with different probability distribution for $\pi_1$

An investor who waits to redeem, in contrast, receives a payoff that depends on when the run is detected and whether she ends up being type 2 or type 3. If the run is detected in period 1, the fund will preemptively liquidate enough investment to serve all investors who redeem in both periods 1 and 2. In this case, no redemption fees are applied and the investor will either receive 1 (in period 2) or $R$ (in period 3). If the run is not detected until period 2, however, redemption fees will be imposed if the realized liquidation value $r_2$ is low. In this case, the investor will receive less than 1 if she ends up being type 2, which may be costly in utility terms.

The red curves in the figure depict the expected value of waiting as a function of the parameter $\delta$, which measures the size of the run. If $\delta$ is large, a run is very likely to be detected in period 1 and waiting is better. If $\delta$ is small, a run is small in size and will lead to little liquidation of investment, so waiting is again better. The danger area for an investor who waits is when $\delta$ is in the intermediate region: small enough that a run might go undetected in period 1, but large enough to lead to a substantial redemption fee in period 2 if $r_2$ is low. In each panel of the figure, the investor's best response is to redeem early – meaning a run equilibrium exists – for a range of values of $\delta$.

In moving from panel (a) to panel (c) in the figure, the expected value of $\pi_1$ increases, which means a run is more likely to be detected in the first period. The figure shows that the set of $\delta$ for which the run equilibrium exists becomes smaller. In this sense, a run equilibrium is most likely to exist when there is more uncertainty about the timing of fundamental withdrawal demand. Panel (d) is based on a uniform distribution for $\pi$ and illustrates that the results do not depend on the particular functional form chosen for $\pi_1$.

**Example 2:** Let $r_1 = \bar{r} = 1$. $\pi_1$ follows a truncated normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.3$. We consider the following four cases:

(a) $\pi = 0.6, R = 1.03, \underline{r} = 0.8, q = 0.5;$

(b) $\pi = 0.6, R = 1.03, \underline{r} = 0.5, q = 0.8;$

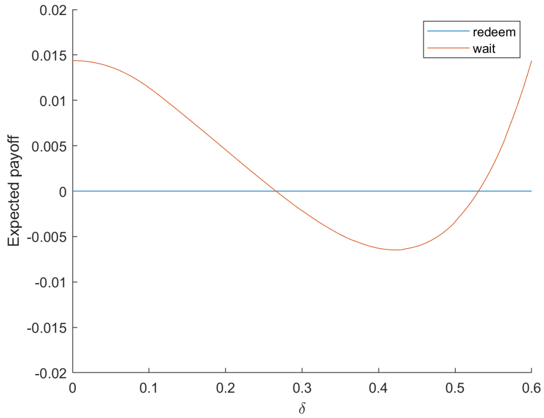
(c) $\pi = 0.7, R = 1.03, \underline{r} = 0.8, q = 0.5;$

(d) $\pi = 0.6, R = 1.01, \underline{r} = 0.8, q = 0.5;$

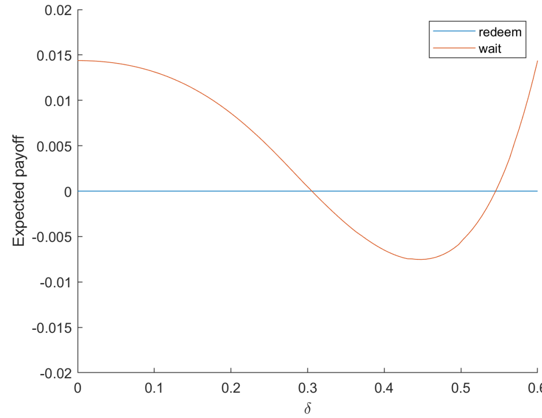
This second example investigates how the probability distribution over the future liquidation value of investment ($r_2$) affects the existence of a run equilibrium. The results are depicted in Figure 2.

Moving from panel (a) to panel (b) in this figure, the downside risk is larger (that is, $\underline{r}$ is much smaller), but the set of $\delta$ for which the fund is fragile is essentially unchanged. In other words, a worse downside in period 2 does not necessarily make the fund more fragile. Comparing panel (a) to panels (c) and (d) shows the fragility set tends to increase as there are more impatient investors or the interest rate of the long-term asset is smaller, at least in the context of this example.

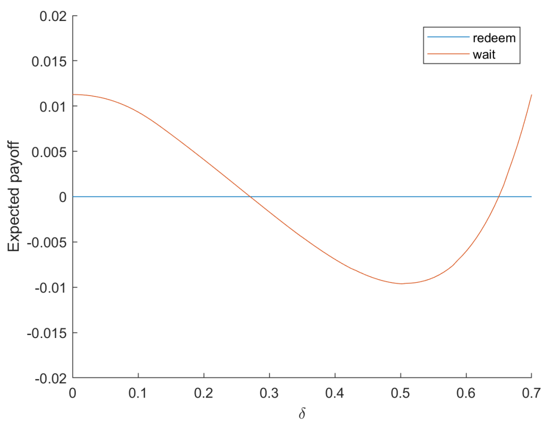
**Example 3:** Let $\pi = 0.6$ and $R = 1.01$. $\pi_1$ follows a truncated normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.3$. In this example, we focus on the scenario where



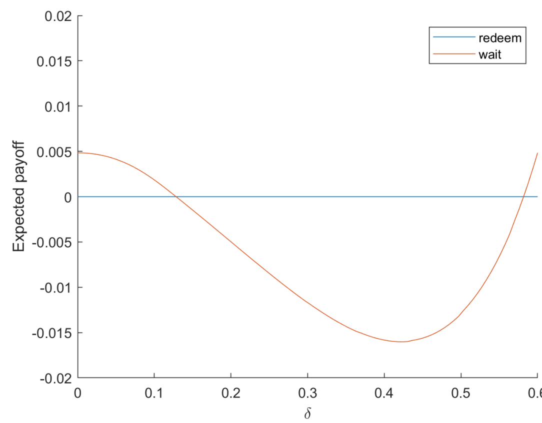

(a) $\pi = 0.6, R = 1.03, \underline{r} = 0.8, q = 0.5$




(b) $\pi = 0.6, R = 1.03, \underline{r} = 0.5, q = 0.8$




(c) $\pi = 0.7, R = 1.03, \underline{r} = 0.8, q = 0.5$



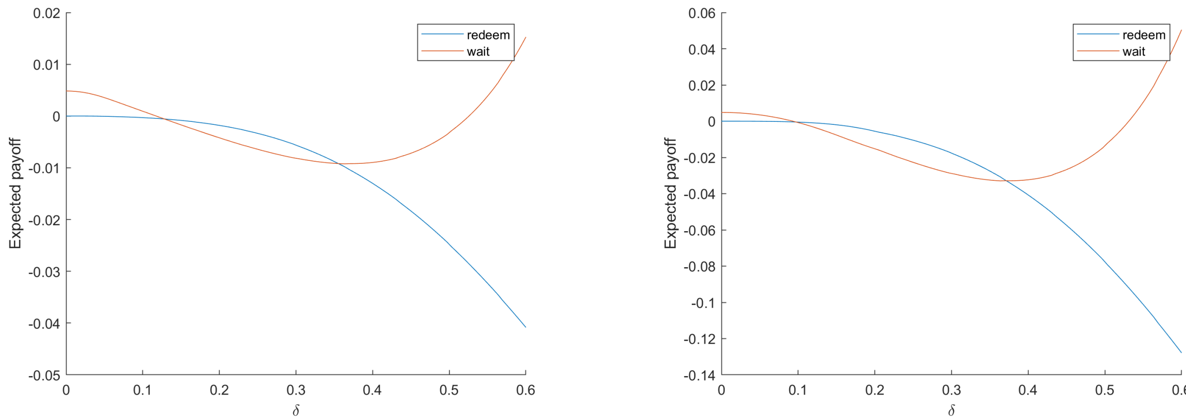

(d) $\pi = 0.6, R = 1.01, \underline{r} = 0.8, q = 0.5$

Figure 2: Fragility region of redemption fees with different set of parameters

$r_1 = \bar{r} < 1$ and $q = 1$, i.e., liquidation value of investment is low in period 1 and will be unchanged in period 2. We consider two cases: $r_1 = 0.9$ and $r_1 = 0.7$. Figure 3 depicts the expected payoff of redeeming and waiting for each case. Because liquidation is now costly in period 1, a redemption fee will be imposed if redemption demand is larger than $\pi$ in this period. As a result, the payoff from redeeming in period 1 is now a decreasing function of $\delta$, as shown in both panels of the figure. In both cases, it is again the case that the fund is fragile for a range of intermediate values of $\delta$. This range is larger for the case where the liquidation value of investment is lower.

### 3.3 Discussion

Engineer (1989) and Cipriani et al. (2014) have shown that a policy of restricting withdrawals




(a) $r_1 = 0.9$

(b) $r_1 = 0.7$

Figure 3: Fragility region of redemption fees with different liquidation values

once demand is unusually high may be ineffective at preventing runs. Our environment with redemption fees and no sequential service gives the fund more flexibility to shape incentives in the redemption game. One might have hoped that such flexibility would allow the fund to implement the efficient allocation without also introducing a run equilibrium. Indeed, our reading of the 2014 reforms is that they were based on precisely this type of reasoning. Policymakers emphasized, for example, that a redemption fee can correct the negative externality that arises when current redemptions leave the fund with a less liquid portfolio.

The examples above show that, in many cases, this approach does not work. The problem arises when $(i)$ investors believe a run is starting but will be small enough that it is not detected by the fund in the current period, and $(ii)$ liquidating investment in future periods may be costly. An attentive investor will then recognize that redemption fees are likely to be larger in the future, creating the incentive to redeem preemptively. This type of incentive appears to have played an important role in the runs on money market funds in March 2020.[19] Put differently, this type of policy corrects the negative externality associated with early redemptions only if redemption demand is large enough to immediately indicate a run is underway, which is not always the case.

What should policymakers do? In the next section, we show that redemption fee policies can be effective at preventing runs if the fees are imposed more aggressively. In particular, a fee must be applied for at least some levels of fundamental withdrawal demand.

[19] For example, the SEC stated in its MMF reform proposal of in February 2022 that "the possibility of an imposition of a fee ... appears to have contributed to incentives for investors to redeem." Also, see Li et al. (2021) for more empirical evidence of this effect.

# 4 Preventing runs

In this section, we study redemption fee policies that deviate from the first-best allocation rule with the goal of eliminating the run equilibrium. We first formulate the problem of finding the best run-proof fee policy. We characterize the solution to this problem and show that it can have surprising features. We then evaluate the 2023 MMF reforms using our framework and show how they differ from the optimal policy. Finally, we discuss the role of portfolio restrictions in the optimal design of run-proof arrangements.

## 4.1 The best run-proof contract

In situations where the policy in Section 3 admits a run equilibrium, preventing runs requires the fund to impose a fee in at least some circumstances where withdrawal demand is in the normal range. It is always possible to prevent runs by setting the fees aggressively enough. If, for example, $c_1(m_1)$ is set to the lowest possible liquidation value of investment in any period, for all $m_1$, then early redemptions by some investors will always increase the payments to investors in subsequent periods, removing the incentive to run. While such a policy prevents runs, it also gives low consumption to type-1 investors in the no-run equilibrium, sharply reducing the fund's attractivess. Our goal in this section is to find the least costly way for the fund to rule out a preemptive run.

We allow the fund to choose any feasible payment function $c_1(m_1)$ in period 1 when redemption demand is consistent with fundamentals, that is, when $m_1$ is in $[0, \pi]$.[20] We continue to focus on the log utility case, which implies the fund will set $c_1(m_1) \leq 1$ for all $m_1$. When $c_1$ is strictly less than 1, meaning a fee is imposed in period 1, the proceeds are divided efficiently among the remaining investors in periods 2 and 3. Other parts of the analysis, including the time-consistency constraints when redemption demand is greater than $\pi$, remain the same as in Section 3. It is straightforward to show that any such policy will again generate a no-run equilibrium in the redemption game. We want to identify the welfare maximizing policy subhec to the constraint that no run equilibrium exists.

When a fee is imposed in period 1, the fund will have extra liquid assets available in period 2. If the fee is relatively small, the fund will find it optimal to use all of those assets to increase the consumption of type 2 investors, since the efficient allocation has $c_2 < c_3$. If the fee imposed in period 1 is sufficiently large, however, the fund will divide the proceed in

[20] Since $\pi$ is known, the total redemption demand in period 2 $m_1 + m_2$ is at least $\pi$. Therefore, this new flexibility is only relevant in period 1.

such a way that the consumption of type 2 and type 3 investors is equalized. In other words, for any $m_1 \leq \pi$ and $c_1(m_1) \leq 1$ the fund's optimal payments in periods 2 and 3 are given by

$$c_2(m_1; c(m_1)) = \min\{c_2^n(m_1; c(m_1)), c_2^e(m_1; c(m_1))\} = \min\{\frac{\pi - m_1 c(m_1)}{\pi - m_1}, \frac{R(1-\pi) + \pi - m_1 c(m_1)}{1 - m_1}\};$$
$$c_3(m_1; c(m_1)) = \max\{c_3^n(m_1; c(m_1)), c_3^e(m_1; c(m_1))\} = \max\{R, \frac{R(1-\pi) + \pi - m_1 c(m_1)}{1 - m_1}\}.$$

The superscript $n$ in these expressions represents the case where the fund uses all of its liquid assets in period 2 (holding no excess liquidity), while $e$ represents the case where the fund holds some liquid assets until period 3 as a way to equalize the consumption of type 2 and 3 investors. Using this notation, the expected utility of investors in the no-run equilibrium for a given policy $c_1(m_1)$ can be written as

$$\int_0^{\pi} [m_1 u(c(m_1)) + (\pi - m_1) u(c_2(m_1; c(m_1))) + (1-\pi) u(c_3(m_1; c(m_1)))] f(m_1) dm_1, \quad (11)$$

where $m_1 = \pi_1$. Note that if $c(m_1) = 1$ for all $m_1$, we would have $c_2(m_1) = 1$, $c_3(m_1) = R$, and the first-best level of welfare would obtain.

To prevent runs, the fund must set the function $c(m_1)$ so that waiting to redeem becomes a dominant strategy for an attentive non-type-1 investor in period 1. In other wrods, a non-type-1 investor who expects all other attentive investors to redeem in period 1 must be willing to wait. This run-proof condition can be written as

$$\int_0^{\frac{\pi-\delta}{1-\delta}} u(c(m_1)) f_n(\pi_1) d\pi_1 + \int_{\frac{\pi-\delta}{1-\delta}}^{\pi} u(r_1(1-\pi) + \pi) f_n(\pi_1) d\pi_1$$
$$\leq \int_0^{\frac{\pi-\delta}{1-\delta}} \{p_{\pi_1}[q u(c_2^l(m_1, m_2; c(m_1), \bar{r})) + (1-q) u(c_2^l(m_1, m_2; c(m_1), \underline{r}))]$$
$$+ (1 - p_{\pi_1})[q u(c_3^l(m_1, m_2; c(m_1), \bar{r})) + (1-q) u(c_3^l(m_1, m_2; c(m_1), \underline{r}))]\} f_n(\pi_1) d\pi_1$$
$$+ \int_{\frac{\pi-\delta}{1-\delta}}^{\pi} [p_{\pi_1} u(r_1(1-\pi) + \pi) + (1 - p_{\pi_1}) u(R(1-\pi) + \frac{R}{r_1}\pi)] f_n(\pi_1) d\pi_1,$$

where

$$c_2^l(m_1, m_2; c(m_1), r_2) = \frac{r_2(1-\pi) + \pi - m_1 c(m_1)}{1 - m_1},$$
$$c_3^l(m_1, m_2; c(m_1), r_2) = \frac{R(1-\pi) + \frac{R}{r_2}[\pi - m_1 c(m_1)]}{1 - m_1},$$

with $m_1 = \pi_1 + \delta(1-\pi_1)$ and $m_2 = (1-\delta)(\pi - \pi_1)$. The first line of the expression above is the expected utility from redeeming. If $\pi_1$ is small enough that the run is not detected in period 1, $m_1$ will be less than $pi$ and redeeming investors will be paid according to the chosen policy $c_1(m_1)$. If $\pi_1$ is larger, the run will be detected right away and redeeming investors will receive an even share of the liquidation value of the fund's assets. If the investor instead decides to wait, there are several possibilities to consider, as shown on the remaining lines. If $\pi_1$ is small, the investor's consumption depends on whether she ends up redeeming in period 2 or 3 and on the realized liquidation value of investment in period 2. If $\pi_1$ is large and the fund detects the run right away, the fund will do whatever liquidation is needed in period 1 and the investor's consumption will be c2 or c3 as given in Proposition 3.

The run-proof condition can be simplified to the following:

$$
\begin{aligned}
B(\delta) \geq \int_0^{\frac{\pi-\delta}{1-\delta}} & \{u(c(m_1)) - p_{\pi_1}[qu(c_2^l(m_1, m_2; c(m_1), 1)) + (1-q)u(c_2^l(m_1, m_2; c(m_1), \underline{r}))] \\
& - (1-p_{\pi_1})[qu(c_3^l(m_1, m_2; c(m_1), 1)) + (1-q)u(c_3^l(m_1, m_2; c(m_1), \underline{r}))]\} f_n(\pi_1) d\pi_1,
\end{aligned}
$$

where

$$
B(\delta) \equiv ln(\frac{R}{r_1}) \int_{\frac{\pi-\delta}{1-\delta}}^{\pi} (1-p_{\pi_1}) f_n(\pi_1) d\pi_1.
$$

Note that $B(\delta)$ is a constant, independent of the choice of contract $c_1(m_1)$ . By change of variables ($m_1 = \pi_1 + \delta(1-\pi_1)$), we can write the run-proof condition as

$$
\begin{aligned}
(1-\delta)B(\delta) \geq \int_\delta^{\pi} & \{u(c(m_1)) - p_{\frac{m_1-\delta}{1-\delta}}[qu(c_2^l(m_1, m_2; c(m_1), \bar{r})) + (1-q)u(c_2^l(m_1, m_2; c(m_1), \underline{r}))] \\
& - (1-p_{\frac{m_1-\delta}{1-\delta}})[qu(c_3^l(m_1, m_2; c(m_1), \bar{r})) + (1-q)u(c_3^l(m_1, m_2; c(m_1), \underline{r}))]\} f_n(\frac{m_1-\delta}{1-\delta}) dm_1.
\end{aligned}
$$

Note that the run-proof condition is only relevant for $m_1 \in [\delta, \pi]$. Therefore, the fund can keep paying investors at par when the redemption demand in period 1 is relatively small, i.e., $m_1 \in [0, \delta)$. This feature is in line with the mandatory fees in the final rule for money market reform announced by SEC in July 2023, whereby a fee becomes mandatory only when current redemptions exceed 5% of the fund's assets. When the redemption demand $m_1$ exceeds $\delta$, the fund needs to compute the optimal $c(m_1)$ that maximizes (11) subject to the run-proof condition. We allow the fund to choose any continuous $c(m_1)$ for $m_1 \in [\delta, \pi]$.

Furthermore, we focus on the following two sets of $c(m_1)$:

$$C^N = \{c(m_1) : \frac{\pi - R(\pi - m_1)}{m_1} \leq c(m_1) \leq 1 \text{ for each } m_1 \in [\delta, \pi]\},$$
$$C^E = \{c(m_1) : \frac{\pi(1-\delta)(1-\underline{r}) + \underline{r}(\delta + m_1)}{(1-\delta)m_1} < c(m_1) < \frac{\pi - R(\pi - m_1)}{m_1} \text{ for each } m_1 \in [\delta, \pi]\}.$$

Note that, if $c(m_1) \in C^N$, the fund chooses the no liquidation payments $(c_2^n(m_1; c(m_1)), c_3^n(m_1; c(m_1)))$ in the no-run equilibrium and theproblem becomes

$$\max_{\{c(m_1)\in C^N\}} \int_\delta^\pi [m_1 u(c(m_1)) + (\pi - m_1)u(\frac{\pi - m_1 c(m_1)}{\pi - m_1}) + (1-\pi)u(R)]f(m_1)dm_1$$
$$s.t. \quad c(m_1) \text{ satisfies the run-proof condition.}$$

Denote this problem as $[P^N]$. If $c(m_1) \in C^E$, the fund chooses the excess liquidity payments $(c_2^e(m_1; c(m_1)), c_3^e(m_1; c(m_1)))$ in the no-run equilibrium. Therefore, the fund's problem becomes

$$\max_{\{c(m_1)\in C^E\}} \int_\delta^\pi [m_1 u(c(m_1)) + (1 - m_1)u(\frac{R(1-\pi) + \pi - m_1 c(m_1)}{1 - m_1})]f(m_1)dm_1$$
$$s.t. \quad c(m_1) \text{ satisfies the run-proof condition.}$$

Denote this problem as $[P^E]$. Note that, if there exists a solution $c^*(m_1)$ to the problem $[P^N]$, $c^*(m_1)$ is the optimal run-proof payment rule. However, if problem $[P^N]$ has no solution, i.e., the run-proof condition is violated for any $c(m_1) \in C^N$, the optimal run-proof payment rule instead sovles problem $[P^E]$.

To being, note that for any $c(m_1) \in C^N$, we have $c(m_1) \geq \frac{\pi - R(\pi - m_1)}{m_1} \geq \frac{\pi - R(\pi-\delta)}{\delta}$. When $R \approx 1$ as in examples 1 and 2, $\frac{\pi - R(\pi - \delta)}{\delta} \approx 1$. Therefore, for any $c(m_1) \in C^N$, $c(m_1) \approx 1$, which violates the run-proof condition since $c(m_1) = 1$ is not run-proof for the given $\delta$. As a result, we can focus on problem $[P^E]$. Let $\mu \geq 0$ be the multiplier for the run-proof condition. Then the optimality condition (Euler-Lagrangian equation) is that for each $m_1 \in [\delta, \pi]$:

$$\underbrace{[\frac{m_1}{c} - \frac{m_1(1-m_1)}{R(1-\pi) + \pi - m_1 c}]f(m_1)}_{\textbf{Marginal cost of reducing welfare}} = \underbrace{\mu\,[\frac{1}{c} + q\frac{m_1}{\bar{r}(1-\pi) + \pi - m_1 c} + (1-q)\frac{m_1}{\underline{r}(1-\pi) + \pi - m_1 c}]f_n(\frac{m_1 - \delta}{1-\delta})}_{\textbf{Marginal benefit of preventing preemptive runs}}. \tag{12}$$

Denote $c^*(m_1, \mu)$ as the solution to (12). If there exists a $\hat{\mu} > 0$ such that the run-proof condition is binding with $c(m_1) = c^*(m_1, \hat{\mu})$ and $c^*(m_1, \hat{\mu}) \in C^E$, $c^*(m_1, \hat{\mu})$ is the solution to problem $P^E$.

Note that (12) highlights the trade-off between welfare maximizing and preventing preemptive runs. In particular, for any $m_1 \in [\delta, \pi]$, when changing the payment $c$ contributes more to the marginal cost of reducing welfare than the marginal benefit of preventing preemptive runs, it is optimal for the fund to impose a smaller redemption fee. To better illustrate this point, we consider those examples studied in section 3 again and choose a $\delta$ that makes the earlier policy subject to a preemptive run. Also, since $R \approx 1$, the solution to problem $[P^E]$ is the optimal run-proof policy. The following examples show that this payment rule can be an increasing function or a nonmonotone function. This pattern contrasts with the usual intuition that the payment rule should be a decreasing function of the redemption demand to prevent runs.

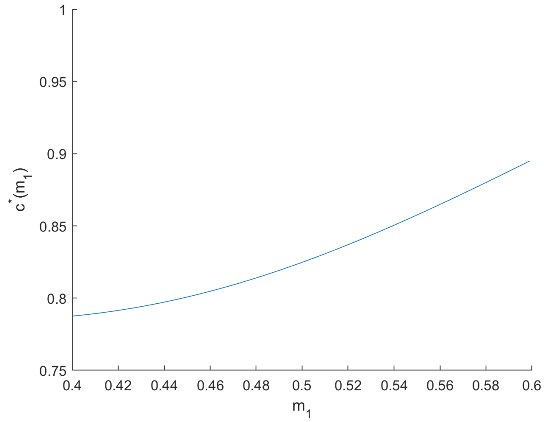
**Example 1 revisited:** Let $\pi = 0.6, R = 1.03, \underline{r} = 0.7, q = 0.5$. Again, we consider those four different probability distributions for $\pi_1$. As shown in example 1, when $\delta = 0.4$, there exists a preemptive run under the original policy for all four probability distributions. Figure 4 depicts the optimal run-proof payment rule for each probability distribution with $\delta = 0.4$. As shown in those figures, the run-proof payment can be an increasing function or a U-shape function of redemption demand.

**Example 3 revisited:** Let $\pi = 0.6$, $r_1 = \bar{r} = r, q = 1$, and $R = 1.01$. $\pi_1$ follows a truncated normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 0.3$. We consider two cases: $r = 0.9$ and $r = 0.7$. As shown in example 3, when $\delta = 0.3$, there exists a preemptive run under the original policy for both cases. Figure 5 depicts the optimal run-proof payment rule for both cases with $\delta = 0.3$. As shown in those figures, the run-proof payment is an increasing function of redemption demand.

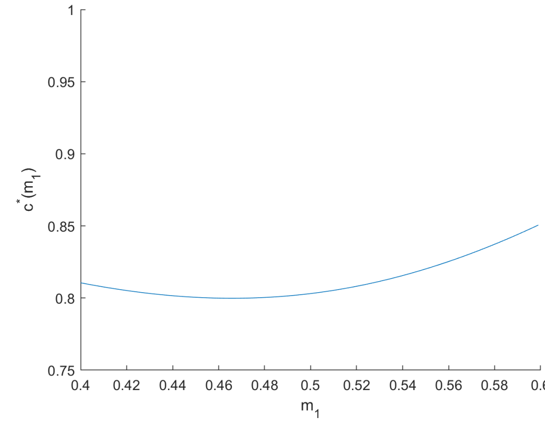
## 4.2 Evaluating the 2023 reforms

In July 2023, the SEC adopted a new set of rules for prime institutional MMFs.[21] The new rules remove funds' ability to impose redemption gates and also eliminate the link between a fund's remaining liquid assets and its ability to impose a redemption fee. Instead, funds are now required to impose a redemption fee in any period in which redemptions exceed 5% of the fund's assets. This threshold is low enough that it is expected to be met with some regularity in normal times. In periods where this threshold is met, a fund is required to impose a redemption fee equal to the cost it would face if it were to sell a pro-rata share

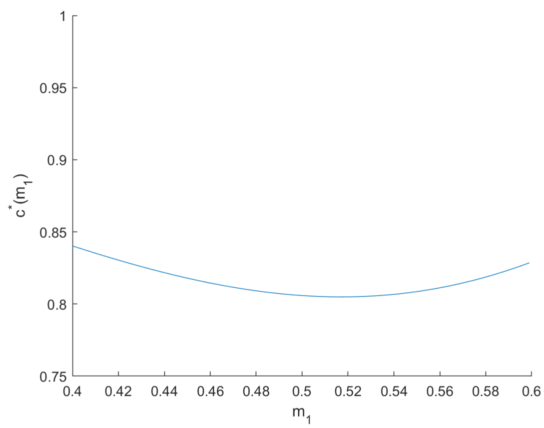
[21] The full text of the 2023 rules is available at https://www.sec.gov/files/rules/final/2023/33-11211.pdf.



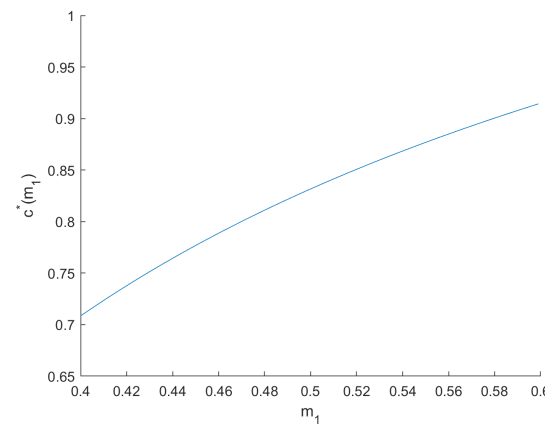

(a) Truncated normal with $\mu = 0$ and $\sigma = 0.3$




(b) Truncated normal with $\mu = 0.3$ and $\sigma = 0.3$




(c) Truncated normal with $\mu = 0.5$ and $\sigma = 0.3$

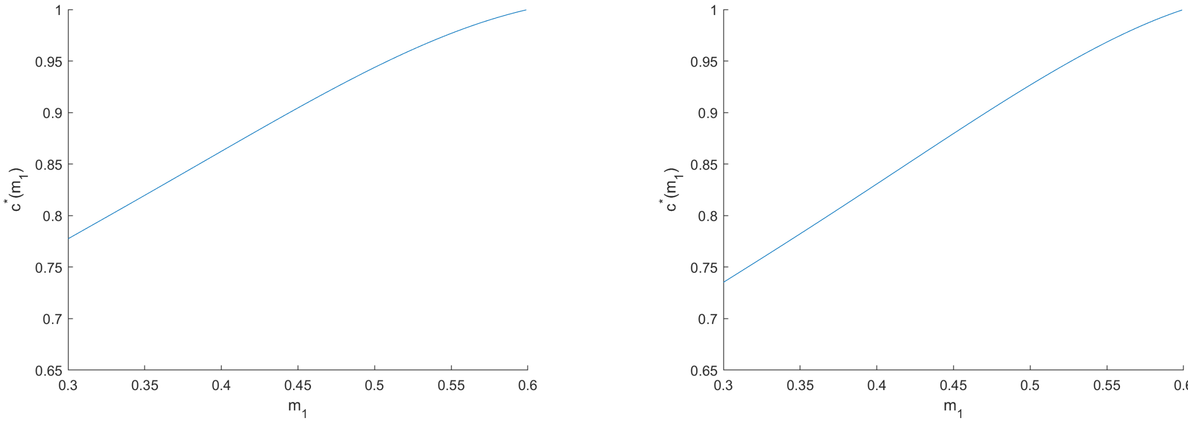



(d) Uniform on $[0, \pi]$

Figure 4: Run-proof payments with different probability distributions for $\pi_1$ and $\delta = 0.4$

of each security in its portfolio. In other words, redeeming investors will receive the current liquidation value of a "vertical slice" of the fund's portfolio.

The new rules are designed to "reduce potential first-mover advantage" in share redemption and thereby prevent runs. However, our analysis above indicates that redemption incentives are complex and forward-looking. We examine the effectiveness of these new rules in our model under two distinct scenarios. In one scenario, investment is currently illiquid ($r_1 < 1$) and market conditions are expected to remain unchanged ($r_2 = r_1$). In the second scenario, market conditions are currently normal ($r_1 = 1$), but investors believe conditions may deteriorate ($\underline{r} < 1$ and $q < 1$). We show that the new rules successfully prevent runs in the first scenario but not in the second.




(a) $r = 0.9$ (b) $r = 0.7$

Figure 5: Run-proof payments with different different liquidation values and $\delta = 0.3$

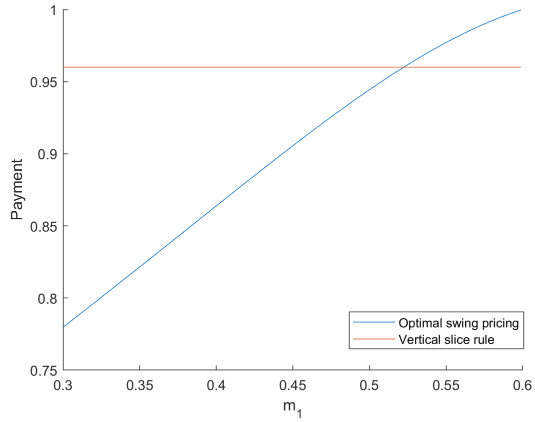
**Secenario 1:** $r_1 = r_2 = r < 1$**.** In scenario 1, the vertical slice rule implies that the fund should set $c_1(m_1) = r(1 - \pi) + \pi$ for $m_1 \in [0, \pi]$. Furthermore, given $c_1(m_1) = r(1 - \pi) + \pi$ for $m_1 \in [0, \pi]$, condition (TC1) requires that, when $m_1 \leq \pi$ and $m_1 + m_2 > \pi$, the fund sets

$$c_2(m_1, m_2, r) = \max\{\frac{\pi - m_1[r(1 - \pi) + \pi]]}{\pi + \delta(1 - \pi) - m_1}, r(1 - \pi) + \pi\}.$$

Therefore, we have $c_2(m_1, m_2, r) \geq r(1 - \pi) + \pi = c_1(m_1)$. In other words, even when the fund does not detect the run in period 1, its optimal payment in period 2 is no less than the payment in period 1. As a result, there is no preemptive run under the vertical slice rule. Figure 6 depicts the optimal run-proof payment rule in example 3 and the corresponding vertical slice rules.

**Secenario 2:** $r_1 = 1; r_2$ **potentially** $< 1$**.** [Refer back to example 1. When $r_1 = 1$, the vertical slice rule imposes no redemption feel while redemption demand is less than $\pi$. So that example shows the new SEC rules are not effective.]

Our results earlier in this section showed that the best run-proof contract has a forward-looking redemption fee: the consumption of type-1 investors in some states depends on the probability distribution for the next-period liquidation value $r_2$. In practice, it may not be straightforward to determine this probability distribution and integrate it into the redemption fee. One might hope to design a simpler redemption fee rule that avoids this complication but is nevertheless run proof. Our example in scenario 2 highlights the problem with this approach. A redemption fee based only on the current liquidation value of investment may not prevent runs in situations when investors fear that market conditions




(a) $r = 0.9$ and $\delta = 0.3$ (b) $r = 0.7$ and $\delta = 0.3$

Figure 6: Optimal run-proof payments and the vertical slice rule in example 3

may rapidly worsen.

## 4.3 Redemption fees vs. portfolio restrictions

So far, we have fixed the fund's initial portfolio to $(\pi, 1-\pi)$ and focused on using redemption fees to eliminate the run equilibrium. In this section, we take the funds' initial portfolio choice into consideration. Suppose that the fund's initial portfolio is $(s, 1-s)$ with $s \geq \pi$. First, let us consider the case with no redemption fees in normal times, i.e., $c(m_1) = 1$ for all $m_1 \in [0, \pi]$. For a given $\delta$, we want to find the smallest $\bar{s}$ such that there is no preemptive run when the fund is not using fees in normal times. In other words, $(\bar{s} - \pi)$ is the least amount of excess liquidity the fund needs to hold in period 0 to prevent preemptive runs in period 1. To find $\bar{s}$, we first derive the time-consistent payment schedules for any $s \in [\pi, 1]$.

**Proposition 4.** *Condition (TC2) requires that, when $m_1 > \pi$, the fund's optimal payment adjustment is divided into three regions in terms of $s$:*

1. *Liquidation region: If $s \in [\pi, \bar{s}_1^D)$, the fund sets $c_1 = c_2(r_2) = r_1(1-s) + s$, and $c_3(r_2) = R(1-s) + \frac{R}{r_1}s$ for any $r_2$, where $\bar{s}_1^D = \frac{r_1[\pi+\delta(1-\pi)]}{r_1[\pi+\delta(1-\pi)+(1-\pi)(1-\delta)]}$;*
2. *No liquidation region: If $s \in [\bar{s}_1^D, \bar{s}_2^D]$, the fund sets $c_1 = c_2(r_2) = \frac{s}{\pi+\delta(1-\pi)}$, and $c_3(r_2) = \frac{R(1-s)}{(1-\pi)(1-\delta)}$ for any $r_2$, where $\bar{s}_2^D = \frac{R[\pi+\delta(1-\pi)]}{R[\pi+\delta(1-\pi)+(1-\pi)(1-\delta)]}$;*
3. *Excess liquidity region: If $s \in (\bar{s}_2^D, 1]$, the fund sets $c_1 = c_2(r_2) = c_3(r_2) = R(1-s) + s$ for any $r_2$.*

Therefore, as the fund holds more and more excess liquidity in period 0, it optimally chooses not to liquidate the long-term asset when $s$ is in the middle range and eventually holds excess liquidity until period 3 when $s$ is large enough. The same logic also applies to time-consistent payment schedules under condition (TC1).

**Proposition 5.** *Condition (TC1) requires that, when $m_1 \leq \pi$ and $m_1 + m_2 > \pi$, the fund's optimal payment adjustment in period 2 given $r_2$ is divided into three regions in terms of $s$:*

1. *Liquidation region: If $s \in [\pi, \bar{s}_1^U(m_1, m_2; r_2))$,*

$$c_2(m_1, m_2; r_2) = \frac{r_2(1-s) + s - m_1}{1 - m_1}, c_3(m_1, m_2; r_2) = \frac{R(1-s) + \frac{R}{r_2}(s - m_1)}{1 - m_1},$$

*where $\bar{s}_1^U(m_1, m_2; r_2) = \frac{r_2 m_2 + (1-m_1-m_2)m_1}{r_2 m_2 + 1 - m_1 - m_2}$;*

2. *No liquidation region: If $s \in [\bar{s}_1^U(m_1, m_2; r_2), \bar{s}_2^U(m_1, m_2)]$,*

$$c_2(m_1, m_2; r_2) = \frac{s - m_1}{m_2}, c_3(m_1, m_2; r_2) = \frac{R(1-s)}{1 - m_1 - m_2},$$

*where $\bar{s}_2^U(m_1, m_2) = \frac{Rm_2 + (1-m_1-m_2)m_1}{Rm_2 + 1 - m_1 - m_2}$;*

3. *Excess liquidity region: If $s \in (\bar{s}_2^U(m_1, m_2), 1]$, $c_2(m_1, m_2; r_2) = c_3(m_1, m_2; r_2) = \frac{R(1-s)+s-m_1}{1-m_1}$.*

With Proposition 4 and 5, the next proposition shows the existence of $\bar{s}$ and further characterizes it under some conditions.

**Proposition 6.** *When $c(m_1) = 1$ for $m_1 \in [0, \pi]$, there exists a $\bar{s} \in [\pi, 1]$ such that there is no preemptive run. Furthermore, if there is no preemptive run when $s = \hat{s} = s_1(0, \pi + \delta(1-\pi); \underline{r})$, $\bar{s} \in [\pi, \hat{s})$ and makes the run-proof condition binding, i.e.,*

$$\int_{\delta}^{\pi} [u(1) - p_{\frac{x-\delta}{1-\delta}} E[u(\frac{r_2(1-\bar{s}) + \bar{s} - x}{1-x})] - (1 - p_{\frac{x-\delta}{1-\delta}}) E[u(\frac{R(1-\bar{s}) + \frac{R}{r_2}(\bar{s} - x)}{1-x})]] f_n(\frac{x-\delta}{1-\delta}) dx = (1-\delta)B(\delta).$$

By Proposition 6, if $\bar{s} \in [\pi, \hat{s})$ and makes the run-proof condition binding, we know that there exists a preemptive run for all $s \in [\pi, \bar{s})$ with no redemption fees in normal times. Therefore, we can follow the same steps in section 4.1 to solve the optimal run-proof payment rule $c^*(m_1, s)$ for each $s \in [\pi, \bar{s})$. Let $W(s)$ denote the highest welfare achieved in the no-run equilibrium given that the fund holds $s$ amount of liquid assets in period 0 and adopts the corresponding optimal run-proof payment rule $c^*(m_1, s)$ in period 1, which is given by

$$W(s) = \int_0^{\delta} [\pi_1 u(1) + (\pi - \pi_1) u(c_2(\pi_1; 1, s)) + (1-\pi) u(c_3(\pi_1; 1, s))] f(\pi_1) d\pi_1$$
$$+ \int_{\delta}^{\pi} [\pi_1 u(c^*(\pi_1, s)) + (\pi - \pi_1) u(c_2(\pi_1; c^*(\pi_1, s), s)) + (1-\pi) u(c_3(\pi_1; c^*(\pi_1, s), s))] f(\pi_1) d\pi_1,$$

where

$$c_2(\pi_1; c, s) = \min\{\frac{s - \pi_1 c}{\pi - \pi_1}, \frac{R(1-s) + s - \pi_1 c}{1 - \pi_1}\};$$
$$c_3(\pi_1; c, s) = \max\{\frac{R(1-s)}{1-\pi}, \frac{R(1-s) + s - \pi_1 c}{1 - \pi_1}\}.$$

Note that, as $s$ increases from $\pi$ to $\bar{s}$, $c^*(m_1, s)$ increases for each $m_1$ since having more liquidity in period 0 reduces the redemption fee in period 1, which benefits type-1 investors. However, as $s$ increases from $\pi$ to $\bar{s}$, $c_3(\pi_1; c, s)$ decreases for each $\pi_1$ and $c$, since holding more liquidity in period 0 means less investment matured in period 3, which makes type-3 investors worse off. The next example illustrates that, under some parameter values, the cost of holding excess liquidity in period 0 outweighs the benefit.

**Example 4:** Let $\pi = 0.6, R = 1.03, r_1 = \bar{r} = 1, \underline{r} = 0.8$, and $q = 0.5$. Furthermore, $\pi_1$ follows the truncated normal distribution in $[0, \pi]$ with the mean equal to 0 and standard deviation $\sigma = 0.3$. As shown in example 2, we pick $\delta = 0.4$ so that there exists a preemptive run under redemption fees. Given those parameter values, we solve $\bar{s} = 0.69$ and Figure 7 depicts $W(s)$ for $s \in [\pi, \bar{s})$, which is a strictly decreasing function. Therefore, in this example, holding excess liquidity reduces social welfare.

# 5 Extensions

The analysis above assumes the fund knows the parameter $\delta$, which measures how many non-type-1 investors would participate if a run were to occur in period 1. In this section, we present two extensions to the analysis that weaken this assumption. First, we allow for $\delta$ to be a random variable with a known distribution. We then consider a robust control approach in which the fund believes that, given its choice of contract, nature will select $\delta$ in a way that maximizes investors' incentive to run. We derive the form of the best run-proof contract in both cases. While the details of the results vary across specifications, the analysis shows how our general approach can be applied for different assumptions about the fund's

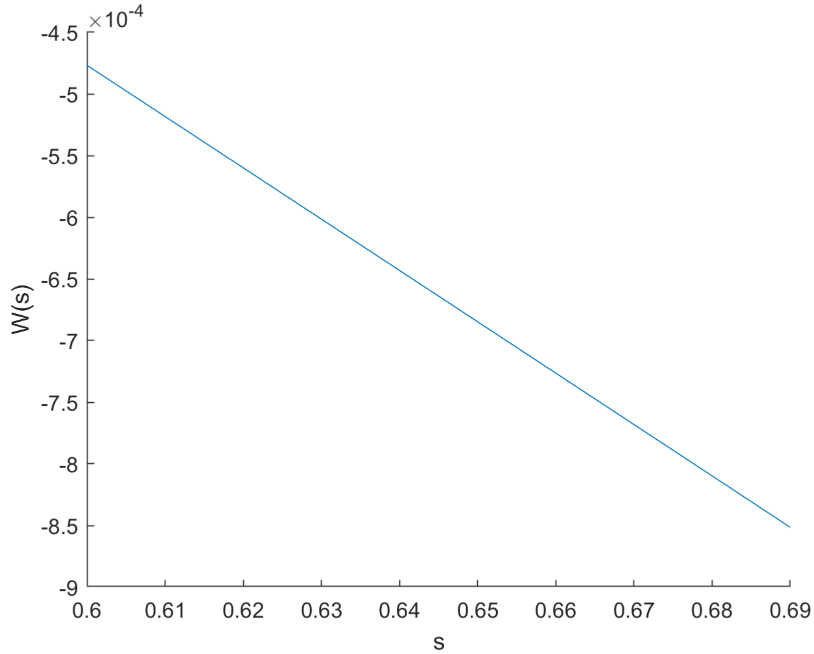



Figure 7: Welfare function $W(s)$

information about the size of a potential run.

## 5.1 Random $\delta$

In this section, we study the optimal run-proof payment rule when $\delta$ is random. More specifically, we assume that $\delta$ takes values in $\{\delta_L, \delta_H\}$, and $p(\delta = \delta_L) = \hat{q}$. Furthermore, we assume that $\delta$ and $\pi_1$ are stochastically independent. For simplicity, let $r_2$ be known and $r_1 = r_2 = r$, where $r \in (\frac{\pi(1-\delta_L)}{\pi+\delta_L(1-\pi)}, 1]$. Note that $\delta$ being random does not change the efficient allocation and condition (TC1). However, with $\delta$ being random, the time consistency condition when $m_1 > \pi$ is more complicated and needs to be divided into two conditions:

(TC2H) For $m_1 > \pi + \delta_L(1-\pi)$, $\{c_1, c_2, c_3\}$ must maximize equation (9) subject to constraints in equations (3) - (5) with $m_2 = \pi + \delta_H(1-\pi) - m_1$.

(TC2L) For $m_1 \in (\pi, \pi + \delta_L(1-\pi)]$, $\{c_1, c_2, c_3\}$ must maximize equation (9) with expectation over $\delta$ subject to constraints in equations (3) - (5) for $m_2 = \pi + \delta_H(1-\pi) - m_1$ and $m_2 = \pi + \delta_L(1-\pi) - m_1$, respectively.

Here, since $\delta$ is random, the fund is uncertain about the exact scale of the run when $m_1$ is possible under both values of $\delta$, i.e., $m_1 \in (\pi, \pi + \delta_L(1-\pi)]$. If $m_1 > \pi + \delta_L(1-\pi)$, a redemption demand that is higher than the largest possible redemption demand with $\delta_L$, the fund knows for sure that it is experiencing a bigger scale run in period 1, i.e., $\delta = \delta_H$.

**Redemption fees.** First, note that the time consistent payment schedule at $t = 2$ when $m_1 \le \pi$ and $m_1 + m_2 > \pi$ stays the same as in 3.2. Following the same analysis in Proposition 3, condition (TC2H) requires that the fund sets $c_1 = c_2 = r(1-\pi) + \pi$ since $r > \frac{\pi(1-\delta_L)}{\pi+\delta_L(1-\pi)}$, i.e., $r(1-\pi)+\pi > \frac{\pi}{\pi+\delta_L(1-\pi)}$. Similarly, condition (TC2L) requires that the fund sets $c_1 = c_2(\delta) = r(1-\pi)+\pi$ for each $\delta$.

Next, we study whether a preemptive run exists under redemption fees that implement the first-best allocation when $\delta$ is random. Same as in section 3.2, we compare a non-type-1 investor's expected payoff of redeeming and waiting, given that all other attentive non-type-1 investors choose to redeem:

$$\begin{aligned}
\text{Redeem: } & \hat{q}\{\int_0^{\frac{\pi-\delta_L}{1-\delta_L}} u(1) f_n(\pi_1) d\pi_1 + \int_{\frac{\pi-\delta_L}{1-\delta_L}}^{\pi} u(r(1-\pi)+\pi) f_n(\pi_1) d\pi_1\} \\
& + (1-\hat{q})\{\int_0^{\frac{\pi-\delta_H}{1-\delta_H}} u(1) f_n(\pi_1) d\pi_1 + \int_{\frac{\pi-\delta_H}{1-\delta_H}}^{\pi} u(r(1-\pi)+\pi) f_n(\pi_1) d\pi_1\} \\
\text{Wait: } & \hat{q}\{\int_0^{\frac{\pi-\delta_L}{1-\delta_L}} [p_{\pi_1} u(c_2(\pi_1,\delta_L)) + (1-p_{\pi_1}) u(c_3(\pi_1,\delta_L))] f_n(\pi_1) d\pi_1 \\
& + \int_{\frac{\pi-\delta_L}{1-\delta_L}}^{\pi} [p_{\pi_1} u(r(1-\pi)+\pi) + (1-p_{\pi_1}) u(R(1-\pi) + \frac{R}{r}\pi)] f_n(\pi_1) d\pi_1\} \\
& + (1-\hat{q})\{\int_0^{\frac{\pi-\delta_H}{1-\delta_H}} [p_{\pi_1} u(c_2(\pi_1,\delta_H)) + (1-p_{\pi_1}) u(c_3(\pi_1,\delta_H))] f_n(\pi_1) d\pi_1 \\
& + \int_{\frac{\pi-\delta_H}{1-\delta_H}}^{\pi} [p_{\pi_1} u(r(1-\pi)+\pi) + (1-p_{\pi_1}) u(R(1-\pi) + \frac{R}{r}\pi)] f_n(\pi_1) d\pi_1\},
\end{aligned}$$

where

$$\begin{aligned}
c_2(\pi_1,\delta) &= \max\{\frac{\pi-\pi_1-\delta(1-\pi_1)}{(\pi-\pi_1)(1-\delta)}, \frac{r(1-\pi)+\pi-\pi_1-\delta(1-\pi_1)}{(1-\pi_1)(1-\delta)}\}; \\
c_3(\pi_1,\delta) &= \min\{\frac{R}{1-\delta}, \frac{R}{r}\frac{r(1-\pi)+\pi-\pi_1-\delta(1-\pi_1)}{(1-\pi_1)(1-\delta)}\}.
\end{aligned}$$

Similar to examples 1 and 2, the next example illustrates that there exists a preemptive run under the redemption fee payment rule for some intermediate values of $\delta_L$ and $\delta_H$.

**Example 5:** Let $\pi = 0.6, r = 0.7, \delta_H = 0.45, \delta_L = 0.3$, and $q = 0.5$. Furthermore, $\pi_1$ follows the truncated normal distribution in $[0, \pi]$ with the mean equal to 0 and standard deviation

$\sigma = 0.3$. Then it is easy to check that the expected payoff of redeeming is larger than that of waiting.

**Optimal policy.** When $\delta$ is random, the objective function, i.e., the welfare in the no-run equilibrium, stays the same. However, the run-proof condition becomes more complicated. Following the same procedure in section 4.1, the run-proof condition can be simplified to the following:

$$\begin{aligned} B &\geq \hat{q} \int_{\delta_L}^{\pi} G(m_1, c(m_1); \delta_L) dm_1 + (1-\hat{q}) \int_{\delta_H}^{\pi} G(m_1, c(m_1); \delta_H) dm_1 \\ &= \int_{\delta_L}^{\delta_H} \hat{q} G(m_1, c(m_1); \delta_L) dm_1 + \int_{\delta_H}^{\pi} [\hat{q} G(m_1, c(m_1); \delta_L) + (1-\hat{q}) G(m_1, c(m_1); \delta_H)] dm_1, \end{aligned}$$

where

$$\begin{aligned} G(m_1, c; \delta) &= \{u(c) - p_{\frac{m_1-\delta}{1-\delta}} u(\frac{r(1-\pi)+\pi-m_1c}{1-m_1}) - (1-p_{\frac{m_1-\delta}{1-\delta}}) u(\frac{R(1-\pi)+\frac{R}{r}(\pi-m_1c)}{1-m_1})\} H(\delta), \\ H(\delta) &= f_n(\frac{x-\delta}{1-\delta}) \frac{1}{1-\delta}, \end{aligned}$$

and the constant $B$ is

$$B = ln(\frac{R}{r}) [\hat{q} \int_{\frac{\pi-\delta_L}{1-\delta_L}}^{\pi} (1-p_{\pi_1}) f_n(\pi_1) d\pi_1 + (1-\hat{q}) \int_{\frac{\pi-\delta_H}{1-\delta_H}}^{\pi} (1-p_{\pi_1}) f_n(\pi_1) d\pi_1].$$

Therefore, the optimal run-proof payment rule satisfies that, for $x \in [\delta_L, \delta_H]$,

$$[\frac{m_1}{c} - \frac{m_1(1-m_1)}{R(1-\pi)+\pi-m_1c}] f(m_1) = \mu\hat{q} [\frac{1}{c} + \frac{m_1}{r(1-\pi)+\pi-m_1c}] H(\delta_L),$$

and, for $x \in (\delta_H, \pi]$,

$$[\frac{m_1}{c} - \frac{m_1(1-m_1)}{R(1-\pi)+\pi-m_1c}] f(m_1) = \mu [\frac{1}{c} + \frac{m_1}{r(1-\pi)+\pi-m_1c}] [\hat{q} H(\delta_L) + (1-\hat{q}) H(\delta_H)].$$

Denote $c^*(m_1, \mu)$ as the solution to the above equations. If there exists a $\hat{\mu} > 0$ such that the run-proof condition is binding with $c(m_1) = c^*(m_1, \hat{\mu})$ and $c^*(m_1, \hat{\mu}) \in C^E$, $c^*(m_1, \hat{\mu})$ is the optimal run-proof payment rule.

The following example shows the graph of the optimal run-proof payment rule $c(m_1)$ when $m_1 \in [\delta_L, \pi]$. As shown in figure 8, the optimal rule is now a step-wise increasing function in $[\delta, \pi]$ since the fund's concern about a potential scale of a run is different in different regions. In particular, as $m_1$ gets larger than $\delta_H$, the fund needs to worry about the possibility of a larger run in period 1 and, therefore, the optimal payment for $m_1 \in (\delta_H, \pi]$ is much smaller than that for $m_1 \in [\delta_L, \delta_H]$. However, the key trade-off between maximizing welfare and preventing runs stays unchanged.

**Example 5 revisited.** Let $\pi = 0.6, r = 0.7, \delta_H = 0.45, \delta_L = 0.3$, and $\hat{q} = 0.5$. Furthermore, $\pi_1$ follows the truncated normal distribution in $[0, \pi]$ with the mean equal to 0 and standard deviation $\sigma = 0.3$. Figure 8 depicts the optimal run-proof payment rule in $[\delta_L, \pi]$.

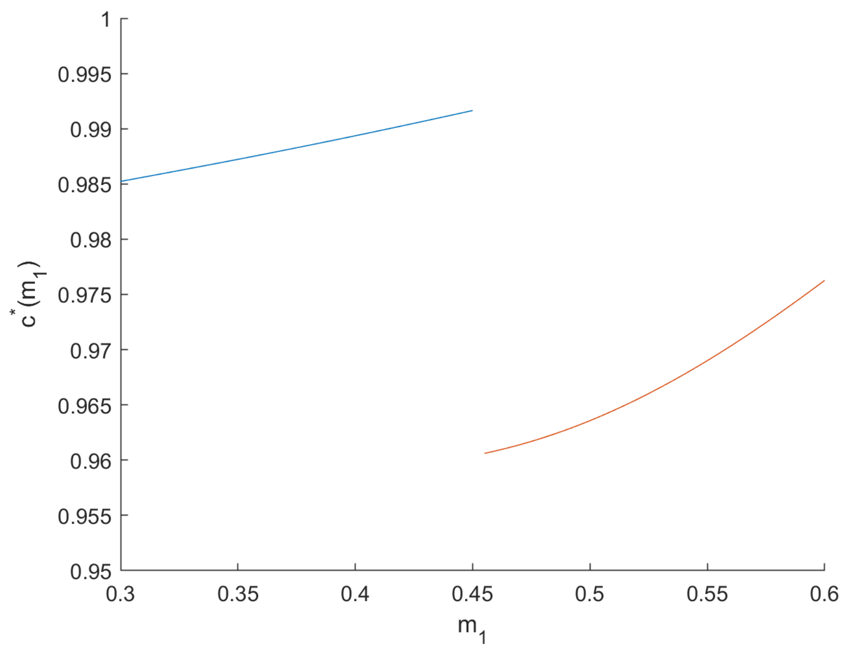



Figure 8: Best run-proof payment rule with random $\delta$

## 5.2 Robust control

So far, we have assumed that the fund has some prior information regarding $\pi_1$, $\delta$, and $r_2$, which allows us to characterize the optimal policy. In this section, we take an adversarial/robust approach to the optimal design of the payment rule. More specifically, we assume that the fund has no information regarding $\pi_1$ and $\delta$ when choosing the payment rule in period 1.[22] Furthermore, we assume that Nature will always move against the fund and

[22] Note that the fund's period 2 payment is chosen ex-post efficiently and does not need any information regarding $\pi_1$ and $\delta$. Furthermore, the fund's period 2 payment rule guarantees that there is no run in period 2.

choose the worst possible distribution over $(\pi_1, \delta, r_2)$ to maximize the fund's exposure to a preemptive run. For simplicity, let the fund's initial portfolio be $(\pi, 1-\pi)$.

**Definition 1.** *A payment rule $c_1(m_1)$ for all $m_1 \in [0,1]$ is **robust run-proof** if it is run-proof for any distribution over $\pi_1$, $\delta$, and $r_2$.*

For any given distribution over $\pi_1$, $F(\pi_1)$, and any payment rule $c_1(m_1)$, we can define the following welfare function:

$$W(c_1(m_1), F(\pi_1)) = \int_0^{\pi} [\pi_1 u(c_1(\pi_1)) + (\pi - \pi_1) u(c_2(\pi_1, \pi - \pi_1; c_1(\pi_1))) \\ + (1-\pi) u(c_3(\pi_1, \pi - \pi_1; c_1(\pi_1)))] dF(\pi_1).$$

**Definition 2.** *A payment rule $c_1^*(m_1)$ for all $m_1 \in [0,1]$ is an **optimal robust run-proof** payment rule if there does not exist a robust run-proof payment rule $c_1'(m_1)$ and a distribution $F(\pi_1)$ such that*

$$W(c_1'(m_1), F(\pi_1)) > W(c_1^*(m_1), F(\pi_1)).$$

To solve the optimal robust run-proof payment rule, let us first consider the following payment rule:

$$c_1(m_1) = \begin{cases} c & \text{if } m_1 = \hat{m}_1 \in [0,1] \\ 0 & \text{otherwise .} \end{cases} \tag{13}$$

To maximize the fund's exposure to a preemptive run, Nature wants to maximize a non-type-1 investor's incentive to run by choosing $\pi_1$ and $\delta$ such that

$$\pi_1 + \delta(1-\pi_1) = \hat{m}_1 \Rightarrow \pi_1 = \frac{\hat{m}_1 - \delta}{1-\delta}.$$

Otherwise, by (13), $c_1(\pi_1 + \delta(1-\pi_1)) = 0$. Therefore, for any $(\pi_1, \delta)$ pair, a non-type-1 investor's incentive to run given that other attentive non-type-1 investors choose to redeem is

$$u(c) - p_{\pi_1} u(c_2(\hat{m}_1, \pi + \delta(1-\pi) - \hat{m}_1; c, r_2)) - (1-p_{\pi_1}) u(c_3(\hat{m}_1, \pi + \delta(1-\pi) - \hat{m}_1; c, r_2)). \tag{14}$$

The next proposition states that, to maximize a non-type-1 investor's incentive to run, Nature chooses $\pi_1 = 0$ and $\delta = \hat{m}_1$, i.e., all withdrawals in period 1 are made by non-type-1 investors, and $r_2 = \underline{r}$.

**Proposition 7.** *For any $c$, $\pi_1 = 0, \delta = \hat{m}_1$, and $r_2 = \underline{r}$ maximizes (14).*

Therefore, it follows from Proposition 7 that a non-type-1 investor's highest incentive to run is

$$u(c) - \pi u(c_2(\hat{m}_1, (1-\hat{m}_1)\pi; c, \underline{r})) - (1-\pi)u(c_3(\hat{m}_1, (1-\hat{m}_1)\pi; c, \underline{r})). \tag{15}$$

Pick $c^*$ such that (15) $= 0$, i.e.,

$$u(c^*) - \pi u(c_2(\hat{m}_1, (1-\hat{m}_1)\pi; c^*, \underline{r})) - (1-\pi)u(c_3(\hat{m}_1, (1-\hat{m}_1)\pi; c^*, \underline{r})) = 0. \tag{16}$$

Note that, for any $c \le c^*$, the following payment rule is a robust run-proof policy.

$$c_1(m_1) = \begin{cases} c & \text{if } m_1 = \hat{m}_1 \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition states that we can construct the optimal robust run-proof payment rule using (16).

**Proposition 8.** *Define, for each $m_1 \in [0,1]$,*

$$c_1^*(m_1) = \min\{c^*(m_1), 1\},$$

*where $c^*(m_1)$ is the solution to equation (16) given $m_1$. Then $c_1^*(m_1)$ for all $m_1 \in [0,1]$ is an optimal robust run-proof payment rule.*

The following numerical example illustrates that $c_1^*(m_1)$ is a decreasing function after $m_1$ is larger than a certain threshold.

**Example 6:** Let $\pi = 0.6, R = 1.03$, and $\underline{r} = 0.8$. Figure 9 depicts the optimal robust run-proof payment rule $c_1^*(m_1)$, which pays at par when $m_1$ is relatively small and then decreases as $m_1$ increases.

# 6 Concluding remarks

This paper studies a mutual fund's optimal design of time-consistent payment rules to prevent preemptive runs. In situations where a run is initially undetected by the fund, we show that

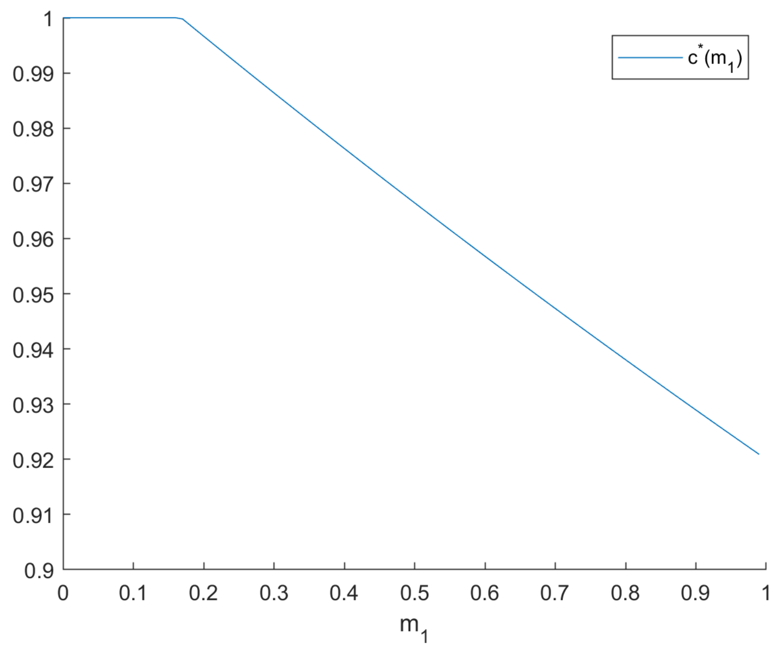



Figure 9: Optimal robust run-proof payment rule

an appropriately designed payment rule can eliminate the first mover advantage while mostly preserving the liquidity function of the fund. In the current setup, an expected deterioration of liquidation value in period 2, i.e., $E[r_2] < 1$, is crucial for the existence of a preemptive run, and the optimal redemption fees can avoid costly liquidation (in period 2) by preventing such a run. If $r_2 \approx 1$, the fund does not need to impose the optimal redemption fees.

One interesting direction for future research is to consider the strategic adoption of the optimal redemption fees among many funds. In particular, suppose that the liquidation value is determined by all funds' liquidation choices. Then, the fund's adoption of the optimal redemption fees can exhibit the property of strategic substitutes. More specifically, if more funds adopt the optimal redemption fees, there will be less costly liquidations in period 2, which will result in a higher liquidation value. Therefore, each fund is less likely to experience a preemptive run and has a weaker incentive to adopt the optimal redemption fees. Such property of strategic substitutes could lead to interesting patterns of the equilibrium adoption of the optimal fees at the market level.

# Appendices

*Proof of Proposition 2.* When $m_1 \leq \pi$ and $m_1 + m_2 > \pi$, condition (TC1) states that the fund's reoptimization problem in period 2 is

$$\begin{aligned} \max \quad & m_2 u(c_2) + (1 - m_1 - m_2) u(c_3) \\ s.t. \quad & m_2 c_2 + e_2 = \pi - m_1 + \ell_2 r_2 \\ & (1 - m_1 - m_2) c_3 = R(1 - \pi - \ell_2) + e_2 \\ & e_2 \geq 0 \\ & \ell_2 \geq 0. \end{aligned}$$

Let $\mu_1, \mu_2, \nu_1, \nu_2$ be the corresponding multipliers. Then, the optimality conditions are

$$u'(c_2) = \mu_1; \tag{17}$$

$$u'(c_3) = \mu_2; \tag{18}$$

$$\mu_2 + \nu_1 = \mu_1; \tag{19}$$

$$r_2 \mu_1 + \nu_2 = R \mu_2; \tag{20}$$

$$\nu_1 e_2 = 0; \tag{21}$$

$$\nu_2 \ell_2 = 0. \tag{22}$$

The following lemma states that the optimal solution always has the fund holding no excess liquidity in period 2, i.e., $e_2 = 0$.

**Lemma 1.** *For any $r_2$, $e_2 > 0$ cannot be optimal.*

*Proof.* Choose any $r_2$. First, consider the case when $e_2 > 0$ and $\ell_2 > 0$, i.e., excess liquidity and extra liquidation in period 2. It follows from (21) and (22) that $\nu_1 = \nu_2 = 0$. By (19) and (20), we have $\mu_2 = \mu_1$ and $r_2 \mu_1 = R \mu_2$, which cannot be satisfied since $r_2 < R$. Therefore, there is no optimal solution in this case.

Next, consider the case when $e_2 > 0$ and $\ell_2 = 0$, i.e., excess liquidity but no extra liquidation. It follows from (21) that $\nu_1 = 0$. By (19), we have $\mu_1 = \mu_2$. Then (17) and (18) imply that $u'(c_2) = u'(c_3)$, i.e., $c_2 = c_3$. As a result, we get

$$c_2 = c_3 = \frac{R(1 - \pi) + \pi - m_1}{1 - m_1}.$$

Then, we know that $e_2 = \pi - m_1 - m_2 c_2 > 0$ if and only if

$$\frac{R(1-\pi) + \pi - m_1}{1 - m_1} < \frac{\pi - m_1}{m_2}. \tag{23}$$

Since $R > 1$ and $m_1 + m_2 > \pi$, (23) never holds. Therefore, there is no optimal solution in this case as well. □

Therefore, by Lemma 1, we can focus on the optimization problem with $e_2 = 0$. Then, given $r_2$, we only need to study two cases: $\ell_2 = 0$ and $\ell_2 > 0$. First, consider the case with $\ell_2 = 0$, i.e., no liquidation of the long-term asset in period 2. The solution, in this case, is straightforward and given by

$$c_2^n(m_1, m_2) = \frac{\pi - m_1}{m_2}, c_3^n(m_1, m_2) = \frac{R(1-\pi)}{1 - m_1 - m_2},$$

where the superscript $n$ represents the case with no liquidation. Next, consider the case with $\ell_2 > 0$, i.e., there is liquidation in period 2. Then we have the following first order condition:

$$r_2 u'(c_2) = R u'(c_3) \Rightarrow c_3 = \frac{R}{r_2} c_2. \tag{24}$$

It follows from the feasibility conditions and (24) that

$$c_2^l(m_1, m_2; r_2) = \frac{r_2(1-\pi) + \pi - m_1}{1 - m_1}, c_3^l(m_1, m_2; r_2) = \frac{R(1-\pi) + \frac{R}{r_2}(\pi - m_1)}{1 - m_1}.$$

Note that $\ell_2 = \frac{1}{r}[m_2 c_2^\ell(m_1, m_2; r_2) - \pi + m_1] > 0$ if and only if

$$c_2^\ell(m_1, m_2; r_2) > \frac{\pi - m_1}{m_2} = c_2^n(m_1, m_2).$$

Therefore, the optimal solution to the fund's problem at $t = 2$ when $m_1 \leq \pi$ and $m_1 + m_2 > \pi$ is the following:

$$c_2(m_1, m_2; r_2) = \max\{c_2^n(m_1, m_2), c_2^l(m_1, m_2; r_2)\},$$
$$c_3(m_1, m_2; r_2) = \min\{c_3^n(m_1, m_2), c_3^l(m_1, m_2; r_2)\},$$

which completes the proof. □

*Proof of Proposition 3.* when $m_1 > \pi$, condition (TC2) states that the fund's problem in period 1 is

$$\begin{aligned}
\max \quad & m_1 u(c_1) + [\pi + \delta(1-\pi) - m_1] E[u(c_2(r_2))] + (1-\pi)(1-\delta) E[u(c_3(r_2))] \\
s.t. \quad & m_1 c_1 + e_1 = \pi + r_1 \ell_1 \\
& [\pi + \delta(1-\pi) - m_1] c_2(r_2) + e_2(r_2) = e_1 + r_2 \ell_2(r_2) \text{ for each } r_2 \\
& (1-\pi)(1-\delta) c_3(r_2) = R(1 - \pi - \ell_1 - \ell_2(r_2)) + e_2(r_2) \text{ for each } r_2 \\
& e_1 \geq 0 \\
& e_2(r_2) \geq 0 \text{ for each } r_2 \\
& \ell_1 \geq 0 \\
& \ell_2(r_2) \geq 0 \text{ for each } r_2.
\end{aligned}$$

Let $\mu_1, \mu_2(r_2), \mu_3(r_2), \nu_1, \nu_2(r_2), w_1, w_2(r_2)$ be the corresponding multipliers, which need to be nonnegative. Then, we have the following first-order necessary conditions:

$$u'(c_1) = \mu_1 \qquad [c_1]$$

$$q u'(c_2(\bar{r})) = \mu_2(\bar{r}) \qquad [c_2(\bar{r})]$$

$$(1-q) u'(c_2(\underline{r})) = \mu_2(\underline{r}) \qquad [c_2(\underline{r})]$$

$$q u'(c_3(\bar{r})) = \mu_3(\bar{r}) \qquad [c_3(\bar{r})]$$

$$(1-q) u'(c_3(\underline{r})) = \mu_3(\underline{r}) \qquad [c_3(\underline{r})]$$

$$\mu_2(\bar{r}) + \mu_2(\underline{r}) + \nu_1 = \mu_1 \qquad [e_1]$$

$$\mu_3(\bar{r}) + \nu_2(\bar{r}) = \mu_2(\bar{r}) \qquad [e_2(\bar{r})]$$

$$\mu_3(\underline{r}) + \nu_2(\underline{r}) = \mu_2(\underline{r}) \qquad [e_2(\underline{r})]$$

$$r_1 \mu_1 + w_1 = R[\mu_3(\bar{r}) + \mu_3(\underline{r})] \qquad [\ell_1]$$

$$\bar{r} \mu_2(\bar{r}) + w_2(\bar{r}) = R \mu_3(\bar{r}) \qquad [\ell_2(\bar{r})]$$

$$\underline{r} \mu_2(\underline{r}) + w_2(\underline{r}) = R \mu_3(\underline{r}) \qquad [\ell_2(\underline{r})]$$

Furthermore, we have the following complementarity slackness [CS] conditions:

$$\nu_1 e_1 = 0, \nu_2(\bar{r}) e_2(\bar{r}) = 0, \nu_2(\underline{r}) e_2(\underline{r}) = 0;$$
$$w_1 \ell_1 = 0, w_2(\bar{r}) \ell_2(\bar{r}) = 0, w_2(\underline{r}) \ell_2(\underline{r}) = 0.$$

To solve this optimization problem, we first establish the following lemmas. The first lemma states that it is never optimal for the fund to hold excess liquidity in period 2.

**Lemma 2.** *For any $r_2$, $e_2(r_2) > 0$ cannot be optimal.*

*Proof.* First, for any $r_2$, $e_2(r_2) > 0$ and $\ell_2(r_2) > 0$ cannot be optimal. To see this, if $e_2(r_2) > 0$ and $\ell_2(r_2) > 0$, it follows from the [CS] conditions and the first order conditions for $e_2(r_2)$ and $\ell_2(r_2)$ that $\mu_2(r_2) = \mu_3(r_2)$ and $r_2\mu_2(r_2) = R\mu_3(r_2)$, which cannot hold at the same time. Therefore, if $e_1 = 0$, $e_2(\bar{r}) > 0$ and $e_2(\underline{r}) > 0$ cannot be optimal since $\ell_2(\bar{r}) > 0$ and $\ell_2(\underline{r}) > 0$ when $e_1 = 0$.

Next, consider the case when $e_2(\bar{r}) > 0$ and $e_2(\underline{r}) = 0$. Since $e_2(\bar{r}) > 0$, we have $\ell_2(\bar{r}) = 0$. It follows from the budget constraints that $c_2(\bar{r}) < c_2(\underline{r})$ and $c_3(\bar{r}) > c_3(\underline{r})$. By the first order condition for $e_2(\bar{r})$, we have $\mu_2(\bar{r}) = \mu_3(\bar{r})$, implying that $c_2(\bar{r}) = c_3(\bar{r})$. Therefore, it follows that $c_2(\underline{r}) > c_3(\underline{r})$, which makes $\nu_2(\underline{r}) < 0$. The same argument applies to the case when $e_2(\underline{r}) > 0$ and $e_2(\bar{r}) = 0$.

Lastly, consider the case when $e_1 > 0, e_2(\bar{r}) > 0, e_2(\underline{r}) > 0$, and $\ell_2(\bar{r}) = \ell_2(\underline{r}) = 0$. By the [CS] conditions and the first order conditions, we have

$$\begin{aligned} \mu_1 &= \mu_2(\bar{r}) + \mu_2(\underline{r}) = \mu_3(\bar{r}) + \mu_3(\underline{r}), \\ r_1\mu_1 + w_1 &= R[\mu_3(\bar{r}) + \mu_3(\underline{r})]. \end{aligned}$$

It follows that $r_1\mu_1 + w_1 = R\mu_1$, implying $\ell_1 > 0$ cannot be optimal in this case. Therefore, we have $\ell_1 = \ell_2(\bar{r}) = \ell_2(\underline{r}) = 0$. Together with the budget constraints, we get $c_1 = c_2(\bar{r}) = c_2(\underline{r}) = c_3(\bar{r}) = c_3(\underline{r}) = R(1-\pi) + \pi > 1$, which contradicts with $e_1 > 0$, $e_2(\bar{r}) > 0$ and $e_2(\underline{r}) > 0$. Therefore, this case cannot be optimal. □

The next lemma states that it is always optimal for the fund to hold excess liquidity in period 1.

**Lemma 3.** *$e_1 = 0$ cannot be optimal.*

*Proof.* By Lemma 2, $e_2(r_2) = 0$ for any $r_2$. Then we are left to check two cases: $\ell_1 > 0$ and $\ell_1 = 0$. If $\ell_1 > 0$, it follows from the [CS] conditions and $\ell_2(r_2) > 0$ that $w_1 = w_2(\bar{r}) = w_2(\underline{r}) = 0$. By the first order conditions for $\ell_1$, $\ell_2(\bar{r})$ and $\ell_2(\underline{r})$, we have

$$r_1\mu_1 = R[\mu_3(\bar{r}) + \mu_3(\underline{r})] = \bar{r}\mu_2(\bar{r}) + \underline{r}\mu_2(\underline{r}). \tag{25}$$

It follows from the first order condition for $e_1$ and (25) that

$$\mu_1 = \nu_1 + \mu_2(\bar{r}) + \mu_2(\underline{r})$$

$$\Rightarrow \frac{\bar{r}}{r_1}\mu_2(\bar{r}) + \frac{\underline{r}}{r_1}\mu_2(\underline{r}) = v_1 + \mu_2(\bar{r}) + \mu_2(\underline{r})$$
$$\Rightarrow r_1 v_1 = (\bar{r} - r_1)\mu_2(\bar{r}) + (\underline{r} - r_1)\mu_2(\underline{r}).$$

Since $\bar{r} > \underline{r}$ and $\ell_2(r_2) > 0$ for any $r_2$, it follows from the budget constraints that $c_2(\bar{r}) > c_2(\underline{r})$, which implies that

$$\begin{aligned} r_1 v_1 &= (\bar{r} - r_1)\mu_2(\bar{r}) + (\underline{r} - r_1)\mu_2(\underline{r}) \\ &= (\bar{r} - r_1) q \frac{1}{c_2(\bar{r})} + (\underline{r} - r_1)(1-q)\frac{1}{c_2(\underline{r})} \\ &< (E[r_2] - r_1)\frac{1}{c_2(\underline{r})} \\ &\leq 0. \end{aligned}$$

The last inequality follows from the assumption that $r_1 \geq E[r_2]$. Therefore, this case cannot be optimal.

Next, consider the case when $\ell_1 = 0$. Similarly, it follows from the [CS] conditions and the first order conditions that

$$\mu_1 = \nu_1 + \mu_2(\bar{r}) + \mu_2(\underline{r}),$$
$$r_1\mu_1 + w_1 = \bar{r}\mu_2(\bar{r}) + \underline{r}\mu_2(\underline{r}).$$

Therefore, we have

$$\begin{aligned} r_1\nu_1 + w_1 &= (\bar{r} - r_1)\mu_2(\bar{r}) + (\underline{r} - r_1)\mu_2(\underline{r}) \\ &= (\bar{r} - r_1) q \frac{1}{c_2(\bar{r})} + (\underline{r} - r_1)(1-q)\frac{1}{c_2(\underline{r})} \\ &< (E[r_2] - r_1)\frac{1}{c_2(\underline{r})} \\ &\leq 0. \end{aligned}$$

Therefore, this case also cannot be optimal. □

By Lemma 2 and 3, we can focus on cases with $e_1 > 0$ and $e_2(\bar{r}) = e_2(\underline{r}) = 0$. In those cases, the next lemma states that it is never optimal for the fund to liquidate in period 2.

**Lemma 4.** *If $e_1 > 0$ and $e_2(\bar{r}) = e_2(\underline{r}) = 0$, $\ell_2(\bar{r}) > 0$ or $\ell_2(\underline{r}) > 0$ cannot be optimal.*

*Proof.* First, consider the case when $\ell_1 > 0$, $\ell_2(\bar{r}) > 0$ and $\ell_2(\underline{r}) > 0$. It follows from the

[CS] conditions that $\nu_1 = w_1 = w_2(\bar{r}) = w_2(\underline{r}) = 0$. By the first order conditions, we have

$$\mu_1 = \mu_2(\bar{r}) + \mu_2(\underline{r}), r_1\mu_1 = \bar{r}\mu_2(\bar{r}) + \underline{r}\mu_2(\underline{r}).$$

Since $\bar{r} \neq r_1 \neq \underline{r}$, the above two equations cannot hold at the same time, meaning this case cannot be optimal.

Next, consider the case when $\ell_1 = 0$, $\ell_2(\bar{r}) > 0$ and $\ell_2(\underline{r}) > 0$. It follows from the [CS] conditions and the first order conditions that

$$w_1 = (\bar{r} - r_1)\mu_2(\bar{r}) + (\underline{r} - r_1)\mu_2(\underline{r}).$$

Since $\bar{r} > \underline{r}$ and $\ell_2(r_2) > 0$ for any $r_2$, we have $c_2(\bar{r}) > c_2(\underline{r})$, which implies that

$$\begin{aligned} w_1 &= (\bar{r} - r_1)q\frac{1}{c_2(\bar{r})} + (\underline{r} - r_1)(1-q)\frac{1}{c_2(\underline{r})} \\ &< (E[r_2] - r_1)\frac{1}{c_2(\underline{r})} \\ &\leq 0. \end{aligned}$$

The last inequality follows from the assumption that $r_1 \geq E[r_2]$. Therefore, this case cannot be optimal.

Next, consider the case when $\ell_2(\bar{r}) = 0$ and $\ell_2(\underline{r}) > 0$. Since $e_2(\bar{r}) = e_2(\underline{r}) = 0$, we have $c_2(\bar{r}) < c_2(\underline{r})$ and $c_3(\bar{r}) > c_3(\underline{r})$. It follows from the first order condition for $\ell_2(\bar{r})$ that

$$\begin{aligned} w_2(\bar{r}) &= R\mu_2(\bar{r}) - \bar{r}\mu_2(\bar{r}) \\ &= Rq\frac{1}{c_3(\bar{r})} - \bar{r}q\frac{1}{c_2(\bar{r})} \\ &< Rq\frac{1}{c_3(\underline{r})} - \bar{r}q\frac{1}{c_2(\underline{r})} = 0. \end{aligned}$$

The last equality follows from the first order condition for $\ell_2(\underline{r})$. Therefore, this case cannot be optimal.

Next, consider the case with $\ell_1 > 0, \ell_2(\bar{r}) > 0$, and $\ell_2(\underline{r}) = 0$. By the [CS] conditions and the first order conditions, we have

$$q\frac{r_1}{c_2(\bar{r})} + (1-q)\frac{r_1}{c_2(\underline{r})} = q\frac{\bar{r}}{c_2(\bar{r})} + (1-q)\frac{R}{c_3(\underline{r})}.$$

Suppose that $\bar{r} = r_1$. Then, it follows that

$$q\frac{r_1}{c_2(\bar{r})} + (1-q)\frac{r_1}{c_2(\underline{r})} = q\frac{r_1}{c_2(\bar{r})} + (1-q)\frac{R}{c_3(\underline{r})} \Rightarrow c_3(\underline{r}) = \frac{R}{r_1}c_2(\underline{r}).$$

Furthermore, we have $c_3(\bar{r}) = \frac{R}{\bar{r}}c_2(\bar{r}) = \frac{R}{r_1}c_2(\bar{r})$. The budget constraints for $\bar{r} = r_1$ are

$$\begin{aligned}[\pi + \delta(1-\pi) - m_1]c_2(\bar{r}) &= \pi + r_1\ell_1 + r_1\ell_2(\bar{r}) - m_1c_1,\\ (1-\pi)(1-\delta)c_2(\bar{r}) &= r_1(1-\pi) - r_1\ell_1 - r_1\ell_2(\bar{r}),\end{aligned}$$

which gives us

$$(1-m_1)c_2(\bar{r}) = r_1(1-\pi) + \pi - m_1c_1.$$

The budget constraints from $\underline{r}$ are

$$\begin{aligned}[\pi + \delta(1-\pi) - m_1]c_2(\underline{r}) &= \pi + r_1\ell_1 - m_1c_1,\\ (1-\pi)(1-\delta)c_2(\underline{r}) &= r_1(1-\pi) - r_1\ell_1,\end{aligned}$$

which gives us

$$(1-m_1)c_2(\underline{r}) = r_1(1-\pi) + \pi - m_1c_1.$$

Therefore, we have $c_2(\bar{r}) = c_2(\underline{r})$, implying $\ell_2(\bar{r}) = 0$ and therefore a contradiction. In other words, when $\bar{r} = r_1$, the case with $\ell_1 > 0, \ell_2(\bar{r}) > 0$, and $\ell_2(\underline{r}) = 0$ cannot be optimal.

Lastly, consider the case with $\ell_1 = 0, \ell_2(\bar{r}) > 0$, and $\ell_2(\underline{r}) = 0$. By the [CS] conditions, we have $\nu_1 = 0$ and $w_2(\bar{r}) = 0$. It follows from the first order conditions for $e_1$ and $\ell_2(\bar{r})$ that

$$\mu_1 = \mu_2(\bar{r}) + \mu_2(\underline{r}), \bar{r}\mu_2(\bar{r}) = R\mu_3(\bar{r}).$$

Since $e_1 > 0$ and $\ell_2(\bar{r}) > 0$, we have $c_2(\bar{r}) > c_2(\underline{r})$ and $c_3(\bar{r}) < c_3(\underline{r})$. Therefore, by the first order condition for $\ell_2(\underline{r})$, we have

$$\begin{aligned}w_2(\underline{r}) &= R\mu_3(\underline{r}) - \underline{r}\mu_2(\underline{r})\\ &= (1-q)[\frac{R}{c_3(\underline{r})} - \frac{\underline{r}}{c_2(\underline{r})}]\\ &< (1-q)[\frac{R}{c_3(\bar{r})} - \frac{\underline{r}}{c_3(\bar{r})}]\end{aligned}$$

$$
\begin{aligned}
&= (1-q)\frac{\bar{r}-\underline{r}}{c_2(\bar{r})} \\
&< (1-q)\frac{\bar{r}-\underline{r}}{c_2(\underline{r})} \\
&= (\bar{r}-\underline{r})\mu_2(\underline{r}).
\end{aligned}
$$

Then, it follows from the first order condition for $\ell_1$ that

$$
\begin{aligned}
w_1 &= R[\mu_3(\bar{r}) + \mu_3(\underline{r})] - r_1\mu_1 \\
&= \bar{r}\mu_2(\bar{r}) + \underline{r}\mu_2(\underline{r}) + w_2(\underline{r}) - r_1\mu_2(\bar{r}) - r_1\mu_2(\underline{r}) \\
&< (\bar{r}-r_1)\mu_2(\bar{r}) + (\underline{r}-r_1)\mu_2(\underline{r}) + (\bar{r}-\underline{r})\mu_2(\underline{r}) \\
&= (\bar{r}-\underline{r})[\mu_2(\bar{r}) + \mu_2(\underline{r})].
\end{aligned}
$$

If $\bar{r} = r_1$, we have $w_1 < 0$, meaning that this case cannot be optimal. In summary, $\ell_2(\bar{r}) > 0$ or $\ell_2(\underline{r}) > 0$ cannot be optimal when $e_1 > 0$ and $e_2(\bar{r}) = e_2(\underline{r}) = 0$. □

By Lemma 2-4, we have $e_1 > 0$ and $e_2(r_2) = \ell_2(r_2) = 0$ for any $r_2$. Therefore, we only need to focus on two cases: $\ell_1 = 0$ and $\ell_1 > 0$. When $\ell_1 = 0$, we have $c_1 = c_2(\bar{r}) = c_2(\underline{r}) = \frac{\pi}{\pi+\delta(1-\pi)}$ and $c_3(\bar{r}) = c_3(\underline{r}) = \frac{R}{1-\pi}$. When $\ell_1 > 0$, we have $c_1 = c_2(\bar{r}) = c_2(\underline{r}) = r_1(1-\pi) + \pi$ and $c_3(\bar{r}) = c_3(\underline{r}) = R(1-\pi) + \frac{R}{r_1}\pi$. □

*Proof of Proposition 4.* First, for any $s$, we still have the usual liquidation case and no liquidation case. In the liquidation case, the fund chooses to pay $c_1 = c_2(r_2) = r_1(1-s) + s$ in both middle periods. In the no liquidation case, the fund chooses to pay $c_1 = c_2(r_2) = \frac{s}{\pi+\delta(1-\pi)}$. If $r_1(1-s) + s > \frac{s}{\pi+\delta(1-\pi)}$, i.e., $s < \frac{r_1[\pi+\delta(1-\pi)]}{(1-\pi)(1-\delta)+r_1[\pi+\delta(1-\pi)]}$, the liquidation case is optimal.

Next, we need to consider another case with excess liquidity. In this case, the fund chooses to pay $c_1 = c_2(r_2) = R(1-s) + s$. In order for this case to be necessary, we need $e_1 > 0$ and $e_2(r_2) > 0$, i.e.,

$$
s - [\pi + \delta(1-\pi)][R(1-s) + s] > 0 \Leftrightarrow s > \frac{R[\pi+\delta(1-\pi)]}{(1-\pi)(1-\delta) + R[\pi+\delta(1-\pi)]} \equiv \bar{s}_2^D.
$$

Let $W^n(s)$ denote the welfare achieved in the no liquidation case and $W^e(s)$ in the excess liquidity case, which are given by

$$
W^n(s) = [\pi+\delta(1-\pi)]ln(\frac{s}{\pi+\delta(1-\pi)}) + (1-\pi)(1-\delta)ln(\frac{R(1-s)}{(1-\pi)(1-\delta)}),
$$

$$W^e(s) = ln(R(1-s)+s).$$

Note that both $W^n(s)$ and $W^e(s)$ are decreasing functions, and $|\frac{dW^n(s)}{ds}| > |\frac{dW^e(s)}{ds}|$. Since $W^n(\bar{s}_2^D) = W^e(\bar{s}_2^D)$, we have $W^n(s) < W^e(s)$ for $s > \bar{s}_2^D$, meaning that the excess liquidity case is optimal. □

*Proof of Proposition 6.* Note that by Proposition 5, when $s > \bar{s}_2^U(m_1, m_2)$, it is optimal for the fund to hold excess liquidity, and $c_2^e(m_1, m_2) = c_3^e(m_1, m_2) > 1$ since $R > 1$. In particular, for the existence of a preemptive run, we are interested in the case with $m_1 = \pi_1 + \delta(1-\pi_1)$ and $m_2 = (1-\delta)(\pi - \pi_1)$, which gives us the following cutoff

$$s_2^U(m_1) = \frac{R[\pi + \delta(1-\pi) - m_1] + (1-\pi)(1-\delta)m_1}{R[\pi + \delta(1-\pi) - m_1] + (1-\pi)(1-\delta)}.$$

Since $\frac{ds_2^U(m_1)}{dm_1} < 0$, we have $s_2^U(m_1) \geq s_2^U(0) = \frac{R[\pi+\delta(1-\pi)]}{R[\pi+\delta(1-\pi)]+(1-\pi)(1-\delta)}$. Therefore, when $s \geq s_2^U(0)$, the fund always hold excess liquidity in period 2, and there does not exist a preemptive run. Note that, as $s$ increases in $[\pi, s_2^U(0)]$, a non-type-1 investor's incentive to run decreases. As a result, there exists a smallest $\bar{s} \in [\pi, s_2^U(0)]$ such that there is no preemptive run. Furthermore, if there is no preemptive run when $s = \hat{s}$, we know that $\bar{s} < \hat{s}$. By Proposition 5, when $s < \hat{s}$, it is optimal for the fund to liquidate the long-term investment for any $m_1$ and $r_2$. Therefore, $\bar{s}$ must make the run-proof condition binding. □

*Proof of Proposition 7.* Note that $\pi_1 = \frac{\hat{m}_1 - \delta}{1-\delta}$ is decreasing in $\delta$, and $p_{\pi_1} = \frac{\pi - \pi_1}{1-\pi_1}$ is also decreasing in $\pi_1$. Therefore, a larger $\delta$ means a larger probability for a non-type-1 investor to be become type-2 in period 2. Furthermore, note that

$$\hat{c}_2(\hat{m}_1, \pi + \delta(1-\pi) - \hat{m}_1; c, r_2) = \max\{\frac{\pi - \hat{m}_1 c}{\pi + \delta(1-\pi) - \hat{m}_1}, \frac{r_2(1-\pi) + \pi - \hat{m}_1 c}{1 - \hat{m}_1}\};$$
$$\hat{c}_3(\hat{m}_1, \pi + \delta(1-\pi) - \hat{m}_1; c, r_2) = \min\{\frac{R}{1-\delta}, \frac{R(1-\pi) + \frac{R}{r_2}(\pi - \hat{m}_1 c)}{1 - \hat{m}_1}\}.$$

Therefore, to maximize (14), Nature chooses $r_2 = \underline{r}$. In the case with no liquidation in period 2, a non-type-1 investor's expected payoff of waiting is given by

$$\begin{aligned} H(\delta; m_1) &= p_{\frac{m_1 - \delta}{1-\delta}} ln(\frac{\pi - \hat{m}_1 c}{\pi + \delta(1-\pi) - \hat{m}_1}) + (1 - p_{\frac{m_1-\delta}{1-\delta}}) ln(\frac{R}{1-\delta}) \\ &= \frac{\pi(1-\delta) - m_1 + \delta}{1 - m_1} ln(\frac{\pi - \hat{m}_1 c}{\pi + \delta(1-\pi) - \hat{m}_1}) + \frac{(1-\pi)(1-\delta)}{1-m_1} ln(\frac{R}{1-\delta}). \end{aligned}$$

It follows that

$$\frac{\partial H(\delta; m_1)}{\partial \delta} = \frac{1-\pi}{1-m_1}[ln(\frac{\pi - \hat{m}_1 c}{\pi + \delta(1-\pi) - \hat{m}_1}) - ln(\frac{R}{1-\delta})] + \frac{1-\pi}{1-m_1}(\frac{1}{R} - 1) < 0.$$

Therefore, to maximize (14), Nature chooses $\delta = \hat{m}_1$ and $\pi_1 = 0$, i.e., all redemptions in period 1 are made by non-type-1 investors. □

*Proof of Proposition 8.* Given the payment rule $c_1^*(m_1)$, Nature will choose $\pi_1 = 0$ and $\delta = \delta^*$ to maximize a non-type-1 investor's incentive to run, where

$$\delta^* \in \underset{\delta}{\operatorname{argmax}}\, u(c_1^*(\delta)) - \pi u(c_2(\delta, (1-\delta)\pi; c_1^*(\delta), \underline{r})) - (1-\pi)u(c_3(\delta, (1-\delta)\pi; c_1^*(\delta), \underline{r})).$$

However, the way we set up $c_1^*(m_1)$ guarantees that

$$u(c_1^*(\delta^*)) - \pi u(c_2(\delta^*, (1-\delta^*)\pi; c_1^*(\delta^*), \underline{r})) - (1-\pi)u(c_3(\delta^*, (1-\delta^*)\pi; c_1^*(\delta^*), \underline{r})) \leq 0.$$

Therefore, the payment rule $c_1^*(m_1)$ is robust run-proof. Next, to show that the payment rule $c_1^*(m_1)$ is also optimal, consider any payment rule $\bar{c}_1(m_1)$ such that

$$W(\bar{c}_1(m_1), F(\pi_1)) > W(c_1^*(m_1), F(\pi_1)) \text{ for some } F(\pi_1).$$

There must exist a $\hat{m}_1$ and $k > 0$ such that

$$\bar{c}_1(\hat{m}_1) = c_1^*(\hat{m}_1) + k.$$

However, $\bar{c}_1(m_1)$ is not a robust run-proof payment rule since Nature can pick $(\pi_1, \delta) = (0, \hat{m}_1)$ and make this payment rule subject to a preemptive run. Therefore, the payment rule $c_1^*(m_1)$ is an optimal robust run-proof payment rule. □

# References


Andolfatto, D. and Nosal, E. (2020). Shadow bank runs. *FRB St. Louis Working Paper 2020-012.*

Andolfatto, D., Nosal, E., and Sultanum, B. (2017). Preventing bank runs. *Theoretical Economics*, 12(3):1003–1028.

Capponi, A., Glasserman, P., and Weber, M. (2020). Swing pricing for mutual funds: Breaking the feedback loop between fire sales and fund redemptions. *Management Science*, 66(8):3581–3602.

Capponi, A., Glasserman, P., and Weber, M. (2022). Swing pricing: Theory and evidence. *Annual Review of Financial Economics*, 15.

Cavalcanti, R. d. O. and Monteiro, P. K. (2016). Enriching information to prevent bank runs. *Economic Theory*, 62(3):477–494.

Chen, Q., Goldstein, I., and Jiang, W. (2010). Payoff complementarities and financial fragility: Evidence from mutual fund outflows. *Journal of Financial Economics*, 97(2):239–262.

Cipriani, M. and La Spada, G. (2020). Sophisticated and unsophisticated runs. *FRB of New York Staff Report 956*.

Cipriani, M., Martin, A., McCabe, P. E., and Parigi, B. M. (2014). Gates, fees, and preemptive runs. *FRB of New York Staff Report 670*.

Diamond, D. W. and Dybvig, P. H. (1983). Bank runs, deposit insurance, and liquidity. *Journal of Political Economy*, 91(3):401–419.

Engineer, M. (1989). Bank runs and the suspension of deposit convertibility. *Journal of monetary Economics*, 24(3):443–454.

Ennis, H. M. and Keister, T. (2009a). Bank runs and institutions: The perils of intervention. *American Economic Review*, 99(4):1588–1607.

Ennis, H. M. and Keister, T. (2009b). Run equilibria in the green-lin model of financial intermediation. *Journal of Economic Theory*, 144(5):1996–2020.

Green, E. J. and Lin, P. (2003). Implementing efficient allocations in a model of financial intermediation. *Journal of Economic Theory*, 109(1):1–23.

Huang, X. (2023). Sophisticated banking contracts and fragility when withdrawal information is public. *Theoretical Economics*, (forthcoming).

Jin, D., Kacperczyk, M., Kahraman, B., and Suntheim, F. (2022). Swing pricing and fragility in open-end mutual funds. *The Review of Financial Studies*, 35(1):1–50.

Lewrick, U. and Schanz, J. F. (2017a). Is the price right? swing pricing and investor redemptions. *BIS Working Paper 644*.

Lewrick, U. and Schanz, J. F. (2017b). Liquidity risk in markets with trading frictions: What can swing pricing achieve? *BIS Working Paper 643*.

Li, L., Li, Y., Macchiavelli, M., and Zhou, X. (2021). Liquidity restrictions, runs, and central bank interventions: Evidence from money market funds. *The Review of Financial Studies*, 34(11):5402–5437.

Ma, Y., Xiao, K., and Zeng, Y. (2022). Bank debt, mutual fund equity, and swing pricing in liquidity provision. *Jacobs Levy Equity Management Center for Quantitative Financial Research Paper, December.*

Parlatore, C. (2016). Fragility in money market funds: Sponsor support and regulation. *Journal of Financial Economics*, 121(3):595–623.

Peck, J. and Shell, K. (2003). Equilibrium bank runs. *Journal of Political Economy*, 111(1):103–123.

Sultanum, B. (2014). Optimal diamond–dybvig mechanism in large economies with aggregate uncertainty. *Journal of Economic Dynamics and Control*, 40:95–102.

Voellmy, L. (2021). Preventing runs with fees and gates. *Journal of Banking & Finance*, 125:106065.

Wallace, N. (1988). Another attempt to explain an illiquid banking system: the diamond and dybvig model with sequential service taken seriously. *Federal Reserve Bank of Minneapolis Quarterly Review*, 12(Fall):3–16.

Wallace, N. (1990). A banking model in which partial suspension is best. *Federal Reserve Bank of Minneapolis Quarterly Review*, 14(Fall):11–23.

Zeng, Y. (2017). A dynamic theory of mutual fund runs and liquidity management. *Working Paper No. 42, European Systemic Risk Board, April.*